COLLISIONS

Elasticity

If we drop a ball of glass upon a marble floor, it rebounds almost to its orginal height but if the same ball is dropped upon a wooden floor it rebounds through a much smaller distance.

If we drop an ivory (subst like bone) ball and a wooden ball from the same height upon a hard floor we will note that they will rebound through different heights (distance). Now the velocities of these balls are same when they reach the ground but since they rebound through different heights, their velocities on leaving the floor must be different.

Again, when a ball strikes against a floor or when two balls of any hard material collide they are slightly compressed and when they tend to recover or restore their original shape, they rebound.

The property of the bodies bodies which causes differences in velocities of rebound and which makes them rebound after collision is called elasticity.

Inelastic Body

If a body does not tend to restore its original shape and does not rebound after collision, it is called an inelastic body.

Direct Collision or Impact

Collision between two bodies is direct if the direction of motion of

each is along the common normal at the point at which they touch.

Oblique Impact or Collision

When the directions of motion of either or both colliding bodies before collision are not along the common normal at the point of contact, then the collision is called oblique collision.

Elastic Collision

If K.E of translation is conserved i.e K.E before collision is equal to K.E after collision, then such a collision is called an elastic collision.

When two objects collide and bounce, the impact between them is elastic.

Inelastic Collision

If K.E of translation is not conserved during a collision, then such a collision is called an Inelastic collision or plastic collision.

If bodies stick together upon collision, then impact is inelastic. In an inelastic collision total initial K.E is not equal to the total final K.E because a part of K.E may appear in other form e.g heat energy, sound energy and a part of energy may be used to generate and to dissipate internal stress waves within the bodies.

Remarks # All collisions between real objects are more or less inelastic except when the objects are very rigid such as billiard balls

(2) Here we will consider some simple cases of of the impact of elastic bodies. We can only discuss the cases of particles in collision with particles or planes and of smooth homogeneous spheres in collision with smooth planes or smooth spheres.

Newton's Law of Restitution

Newton found by an experiment that

(1) # If two bodies collide directly, their relative velocity after impact is in a constant ratio to their relative velocity before impact and is in the opposite direction.

(2) If the bodies collide obliquely, their relative velocity resolved along common normal after impact is a in constant ratio to their relative velocity before impact resolved in the same direction and is of opposite sign.

This constant ratio is called co-efficient of elasticity, Restitution or Resilience & is denoted by e

By ① & ② we can define e as

$$\frac{\text{Separation speed along the line of action of impulse}}{\text{Approach speed along the line of action of impulse}} = -e$$

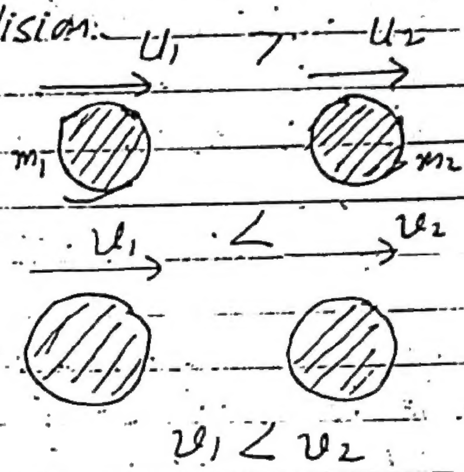
-ve sign shows that relative speed of separation is opposite to the relative speed of approach.

Co-efficient of Restitution for Direct Collision

When two objects are in direct collision (elastic) the speed with which they seperate is usually less than the speed of approach before impact.

Suppose two smooth spheres of masses m_1, m_2 and moving in the same straight line in same direction with velocities u_1, u_2 ~~and~~ collide and v_1, v_2 be their velocities after collision.

Let $u_1 > u_2$. Then v_1 must be less than v_2.



Co-efficient of restitution is

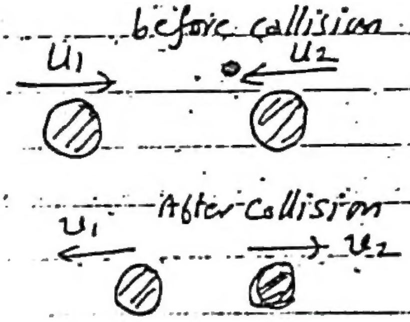
$$\frac{v_1 - v_2}{u_1 - u_2} = -e$$

OR

$$\frac{v_2 - v_1}{u_1 - u_2} = e$$

Note that velocities are measured algebraically i.e all velocities in one direction are taken as +ve while those in opposite direction are taken as -ve.

Thus if the smooth spheres of masses m_1 & m_2 are moving with velocities u_1 & u_2 ($u_1 > u_2$) in opposite sense and collide directly and v_1, v_2 are their velocities after collision in opposite directions as shown, then taking velocities towards right hand side as +ve, we have



$$\frac{-v_1 - v_2}{u_1 - (-u_2)} = -e$$

or

$$\frac{v_1 + v_2}{u_1 + u_2} = e$$

Here u_1 & v_2 are +ve & u_2, v_1 are -ve.

Note #. In problems on elastic impact, a particle A may catch up and collide with, another particle B moving less quickly in the same direction. In this situation A is said to overtake B.

Co-efficient of Restitution for Oblique Collision

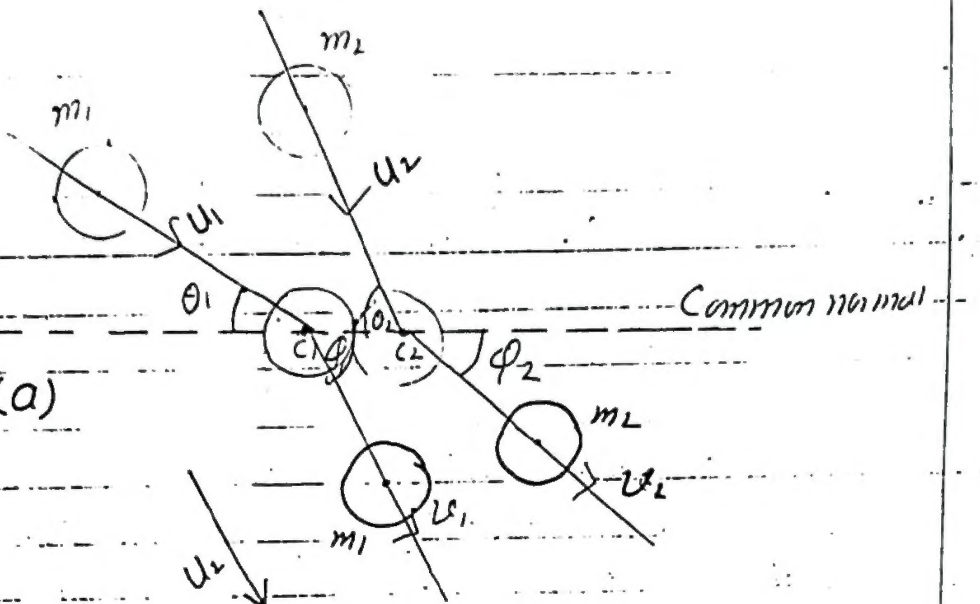


fig (a)

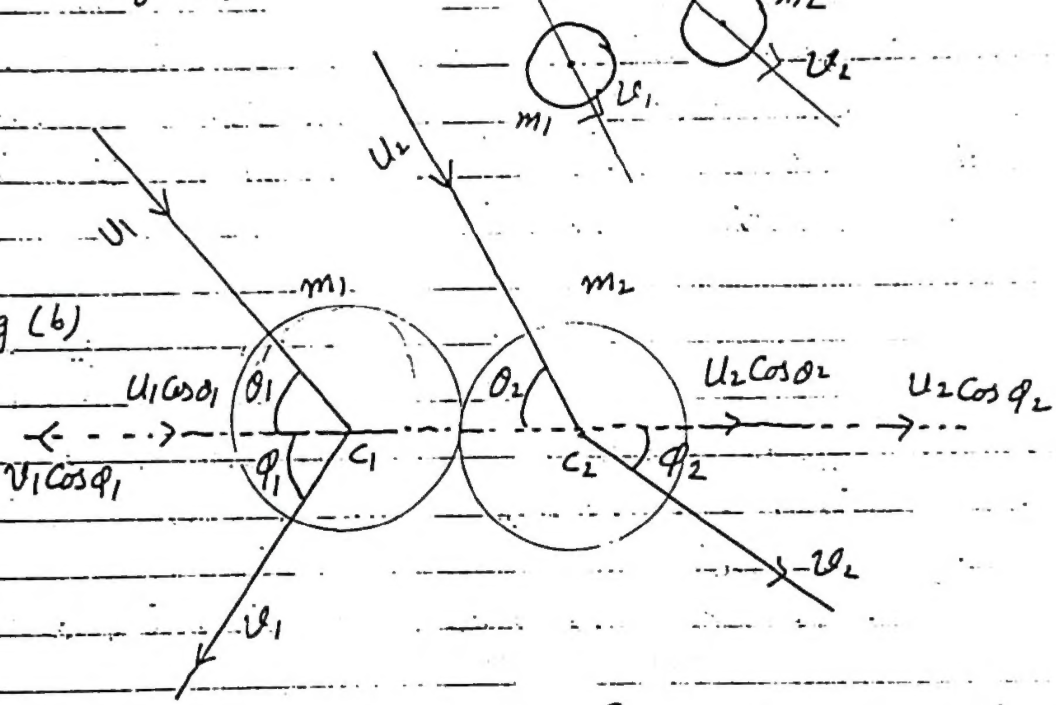


fig (b)

Suppose two smooth spheres of masses m_1, m_2 have initial velocities u_1, u_2 and approach each other on collision course. Let θ_1, θ_2 be angles of direction of motion with normal at the contact before collision. Let v_1 and v_2 be velocities after collision making angles φ_1 and φ_2 with common normal at the contact.

Taking velocities towards right +ve, we have

For fig (a)

$$\frac{v_1 \cos\varphi_1 - v_2 \cos\varphi_2}{u_1 \cos\theta_1 - u_2 \cos\theta_2} = -e$$

OR

$$\frac{v_2 \cos\varphi_2 - v_1 \cos\varphi_1}{u_1 \cos\theta_1 - u_2 \cos\theta_2} = e$$

For fig (b)

$$-\frac{v_1\cos\varphi_1 - v_2\cos\varphi_2}{u_1\cos\theta_1 - u_2\cos\theta_2} = -e$$

$$\Rightarrow \quad \frac{v_1\cos\varphi_1 + v_2\cos\varphi_2}{u_1\cos\theta_1 - u_2\cos\theta_2} = e$$

Value of "e" For Perfectly Elastic Collision

If relative speeds of the colliding bodies do not change i.e relative velocity before collision is equal to relative velocity after collision, then collision is perfectly elastic and this case $e = 1$

Value of "e" For an Inelastic Collision

Impact between objects (between) which do not bounce after collision but Coalesc or stick together is perfectly inelastic. In this case relative speed of seperation is zero and hence $e = 0$ in this case

Value of "e" In General

In general we have

$0 \leq$ |Relative speed of seperation| $\leq$ |Relative speed of approach|

$$\Rightarrow 0 \leq \frac{|\text{Relative speed of seperation}|}{|\text{Relative speed of approach}|} \leq 1$$

$$\Rightarrow 0 \leq e \leq 1$$

Thus numerical value of e lies in the interval $[0, 1]$.

Remarks # (1) The co-efficients of restitution is often considered a constant for given geometries and a

a given combination of contacting materials. However, more careful modern experiments have shown that e also depends also (dep) on velocity of approach and it decreases very slightly for very large velocities of approach of bodies and it approaches unity as the impact velocity approches zero. Because when relative velocity of impact approaches zero, then no impact will occur and bodies will move with same velocities as before collision (we can say) and so $e = 1$.

(2)# It should be noted that a co-efficient of restitution must be associated with a pair of contacting bodies.

Motion of Colliding Bodies along and Perpendicular to the Line of impact i.e normal at Contact And Conservation of Momentum

In general, when an object is struck, the impulse it receives has zero component perpendicular to the direction of blow and hence there is no force in this direction and the momentum of an object remain unchanged perpendicular to the impulse it receives. But in any other direction there is a non-zero impulse component and the increase in momentum in that direction is equal the impulse component in the same direction.

When objects that are free to move, collide, the impulses (and hence the contact forces) are equal and oposite so that the total momentum in any direction remain constant.

When two smooth bodies collide the impulses act along the line joining their centres. Hence there is no force along the perpendicular to the common normal and consequently, no change of velocity in that direction. If the impact not overly severe and if the spheres are highly elastic, they will restore

their original shape after collision. With a more severe impact and with less elastic bodies a permonent deformation may result.

Because the Contact forces (impulsive forces) are equal and opposite along common normal at contact during impact, therefore linear momentum of the System remain unchanged in this direction.

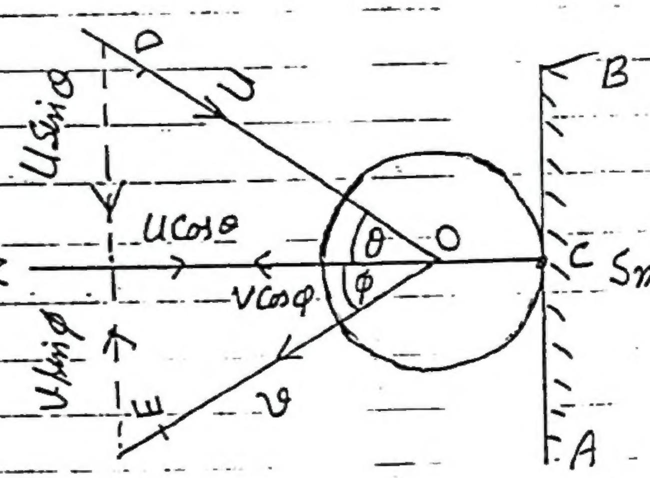
We assume that any forces acting on the sphere during impact, other than large internal impact forces are relatively small and produce negligible impulses compared with the impulse produced by each contact force. Also we assume that no appreciable change in the positions of mass centres occurs during the short duration of impact

Remarks #(1) Direct impact can occur between two moveable objects or between one moveable and one fixed. In both cases the Law of restitution is valid.

(2)# The principle of conservation of linear momentum applies to impact between moveable objects (equal and opposite internal impulses) but not when one of the objects in collision is fixed. (external impulse)

(3)# In considering impact of elastic bodies, we suppose that they are smooth, so that the mutual action between them takes place only in the direction of their common normal at the point of contact and their is no force in the direction perpendicular to the common normal i.e along common tanget.

Impact of a Smoth Sphere with a Smooth Fixed Plane



Smooth fixed plane

Suppose a smooth sphere moving with velocity u strikes a smooth plane AB in a direction making angle θ with the normal to plane at point of contact C. Suppose the ball rebounds with velocity v making angle ϕ with the normal at point C. DO and OE are the directions of motion of the sphere before and after the collision.

The angles θ and ϕ are sometimes called angles of incidence and reflection respectively.

We discuss the subsequent motion of the ball.

Since the plane is smooth, therefore there is no force parallel to the plane (common tangent) and hence the velocity components along this direction are unchanged.

therefor

$$V\sin\phi = U\sin\theta \longrightarrow (1)$$

By Law of restitution, we have

$$\frac{-v\cos\phi - 0}{u\cos\theta - 0} = \frac{-v\cos\phi - 0}{+u\cos\theta - 0} = -e$$

$$\Rightarrow V\cos\phi = eU\cos\theta \longrightarrow (2)$$

(note that velocities in one direction are taken as +ve and in the opposite direction are taken -ve. in present $u\cos\theta$ is +ve & $v\cos\varphi$ is -ve).

$$v\cos\varphi = eu\cos\theta \longrightarrow ②$$

Velocity of ball after impact #

Squaring and adding ① & ②, we get

$$v^2(\sin^2\varphi + \cos^2\varphi) = u^2\sin^2\theta + e^2u^2\cos^2\theta$$

$$v^2 = u^2[\sin^2\theta + e^2(1-\sin^2\theta)]$$

$$= u^2[(1-e^2)\sin^2\theta + e^2] \longrightarrow ③$$

Direction of ball after impact #

Dividing ② by ①, we have

$$\cot\varphi = e\cot\theta \longrightarrow ④$$

This equation gives angle of reflection or direction of ball after impact.

Deductions

Result-I # When the impact is direct i.e ball moves along common normal, then

$$\theta = 0$$

from ④

$$\cot\phi = e\cot 0 = \infty$$

$$\varphi = \cot^{-1}(\infty) = 0$$

i.e. ball moves along common normal after impact

from ③ putting $\theta = 0$

$$v^2 = u^2[(1-e)0 + e^2] = e^2u^2$$

$$v = eu$$

i.e ball moves with velocity eu after impact.
Thus when a smooth ball moving with velocity u impinges directly on a smooth wall or plane, it rebounds in the reverse direction with velocity eu.

Result 2 # If the sphere is perfectly elastic, then $e = 1$

From ③ $v^2 = u^2[(1-1)\sin^2\theta + 1]$

$$v^2 = u^2$$

$$v = u$$

i.e The ball rebounds with same velocity.

From ④ putting $e = 1$

$$\cot\phi = \cot\theta$$

$$\Rightarrow \phi = \theta$$

i.e angle of reflection is same as the angle of incidence.

Thus when a perfectly smooth ball collide obliquely with a smooth fixed plane it rebounds with same velocity and the angle of reflection is equal to the angle of incidence.

Result # If the sphere is perfectly inelastic, then $e = 0$ and

From ③

$$v^2 = [(1-0)\sin^2\theta + 0]u^2$$

$$v = u\sin\theta$$

i.e velocity of ball after collision is equal to the its velocity component along tangent before collision.

putting $e = 0$ in ④

$$\cot\phi = e\cot\theta = 0\cdot\cot\theta = 0$$

$$\Rightarrow \phi = \frac{\pi}{2}$$

i.e ball moves along the plane and does not rebound.

Thus when a smooth ball impinges obliquely on a perfectly inelastic fixed plane, it does not rebound but slides along the and its velocity parallel to the plane remain unchanges.

Loss in K.E of the ball

Let m be the mass of sphere

Initial $K.E = \frac{1}{2}mu^2$

Final $k.E = \frac{1}{2}mv^2$

Loss in $k.E = -\frac{1}{2}mv^2 + \frac{1}{2}mu^2$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

using $v^2 = u^2[(1-e^2)\sin^2\theta + e^2]$ from (4)

$$\text{Loss in } k.E = \frac{1}{2}mu^2 - \frac{1}{2}m\left[u^2\{(1-e^2)\sin^2\theta + e^2\}\right]$$

$$= \frac{1}{2}mu^2 + \left(-\frac{1}{2}mu^2 + \frac{1}{2}me^2u^2\right)\sin^2\theta - \frac{1}{2}me^2u^2$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2\sin^2\theta + \frac{1}{2}me^2u^2\sin^2\theta - \frac{1}{2}me^2u^2$$

$$= \frac{1}{2}mu^2(1-e^2) - \frac{1}{2}mu^2\sin^2\theta(1-e^2)$$

$$= \frac{1}{2}mu^2(1-e^2)(1-\sin^2\theta)$$

$$= \frac{1}{2}mu^2(1-e^2)\cos^2\theta$$

$$= \frac{1}{2}m(u\cos\theta)^2(1-e^2)$$

$$= (\text{Initial } k.E \text{ along normal})(1-e^2) \longrightarrow (5)$$

We not that the loss of K.E is along the normal direction because velocities change

in this direction and there is no change in velocities in direction along the fixed plane

If collision is perfectly elastic, then $e = 1$ and

$$\text{Loss of K.E} = \frac{1}{2}m(u\cos\theta)^2(1-1) = 0$$

i.e there is no loss of K.E in perfectly elastic collision.

If collision is perfectly inelastic, then $e = 0$ and Loss in K.E is maximum given by

$$\text{Loss in K.E} = \frac{1}{2}m(u\cos\theta)^2(1-0)$$

$$= \frac{1}{2}mu^2\cos^2\theta$$

Thus we conclude that their is always loss in K.E of the ball unless the collision is perfectly elastic in which there is no loss in K.E

Impulse of blow or of the force of impact on Ball#

The impulse of the force of contact on the plane is equal (to) and opposite to the impulse of the force of impact on the sphere and is therefore measured by the change of the momentum of the sphere perpendicular to the plane.

Hence the impulse of the blow

= Change in momentum of the sphere in the direction of common normal

$= \cancel{mv\cos\phi - m(}$

$$= mu\cos\theta - m(-v\cos\phi)$$

$$= mu\cos\theta + mv\cos\phi$$

But by ② $v\cos\phi = eu\cos\theta$

$$\text{Impulse of blow} = mu\cos\theta + meu\cos\theta$$

$$= m(1+e)u\cos\theta$$

$= (1+e)\,mu\cos\theta$

If the collision is perfectly elastic, then $e=1$ and impulse of blow is maximum given by

$$\text{Implse of blow} = 2mu\cos\theta$$

If the collision is perfectly inelastic, then $e=0$, the impulse of blow is minimum for a given angle of incidence θ.

$$\text{Min impulse of blow} = mu\cos\theta$$

For direct collision impulse of blow is given by putting $\theta = 0$

$$\text{Impulse of blow for direct collision} = m(1+e)u$$

If $\theta = 90°$ i.e ball impinges along the fixed plane, then ~~impact~~ impulse of blow is

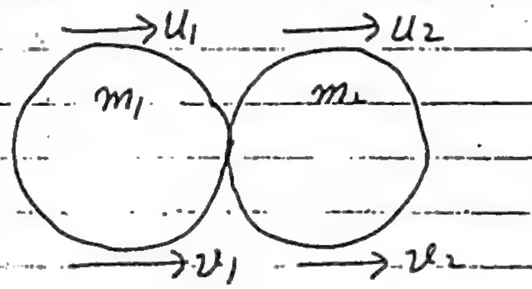
$$\text{Impulse of the blow} = m(1+e)\cos 90°$$
$$= 0$$

i.e there is no impulse of the blow along the fixed plane

Direct Impact

Two elastic spheres of masses m_1, m_2 moving with velocities u_1, u_2 imping directly, find their velocities after impact. If e is co-efficient of restitution.

Suppose that two smooth spheres of masses m_1, m_2 moving with velocities u_1, u_2 collide directly and v_1, v_2 are their velocities after impact.



By Law of restitution.

$$v_1 - v_2 = -e(u_1 - u_2) \rightarrow (1)$$

By Law of Conservation of linear momentum

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \longrightarrow (2)$$

Multiplying (1) by m_2 and adding to (2)

$$(m_1+m_2)v_1 = (m_1 - em_2)u_1 + m_2(1+e)u_2$$

$$v_1 = \left(\frac{m_1 - em_2}{m_1+m_2}\right)u_1 + \frac{m_2(1+e)u_2}{m_1+m_2} \longrightarrow (3)$$

Multiplying (1) by m_1 and subtracting from (2)

$$(m_1+m_2)v_2 = m_1(1+e)u_1 + (m_2 - em_1)u_2$$

$$v_2 = \frac{(m_2 - em_1)u_2}{m_1+m_2} + \frac{m_1(1+e)u_1}{m_1+m_2} \longrightarrow (4)$$

Equations (3) & (4) give velocities after impact.

If the 2nd sphere be moving in a opposite direction to that of the 1st, we must change the sign of u_2

Deductions#

Result#1 If $m_1 = m_2$ and $e = 1$

Then from (3)

$$v_1 = u_2$$

from (4) $\quad v_2 = u_1$

Thus if two equal perfectly elastic balls or spheres moving with velocities u_1, u_2 collide directly, then they interchange their ~~m~~ velocities after impact.

Result#2 Let $m_2 >> m_1$ and $u_2 = 0$ i.e a very very ~~heavy~~ small sphere collide with a ~~small~~ heavy sphere at rest. Then we can ignore mass m_1

as compared to mass m_2

from ③ putting $m_1 = 0$, $u_2 = 0$

$$v_1 = \frac{0 - em_2}{0 + m_2} u_1 + \frac{m_2(1+e)}{0 + m_2}(0)$$

$$v_1 = -eu_1$$

⇒ Spheres of mass m_1 rebounds with velocity eu_1

from ④ putting $m_1 = 0$ $u_2 = 0$

$$v_2 = 0 + 0$$

$$v_2 = 0$$

i.e heavy sphere remains at rest.

Thus when a very small elastic sphere collide with a very heavy elastic sphere, then the small sphere rebounds with vel e times of its initial vel while heavy sphere remain at rest.

<u>Result #2</u> Let $m_1 \gg m_2$ and $u_2 = 0$.

i.e when a very very heavy sphere collide with a small sphere at rest.

from ③ putting $m_2 = 0$ $u_2 = 0$

$$v_1 = \frac{m_1}{m_1} u_1 = u_1$$

i.e heavy sphere moves with same velocity and in same direction.

from ④ putting $m_2 = 0$ $u_2 = 0$

$$v_2 = 0 + \frac{m_1(1+e)u_1}{m_1}$$

$$v_2 = (1+e)u_1$$

i.e the sphere at rest moves with a velocity which is $(1+e)$ times of the velocity of heavy moving sphere.

Result: 4 Let $m_1 = m_2$ and $u_2 = 0$.
i.e. two equal spheres collide and one of which is at rest.

from ③

$$v_1 = \frac{m_2 - em_2}{m_2 + m_2} u_1 + 0$$

$$= \frac{1}{2}(1-e)u_1$$

⇒ moving sphere moves with less velocity in the same direction.

from ④

$$v_2 = \frac{m_2(1+e)u_1}{m_2+m_2} = \frac{1}{2}(1+e)u_1$$

The sphere at rest moves with a velocity which is $\frac{1}{2}(1+e)$ times of the velocity of moving sphere.

Result # 5 Impulse of the blow on the sphere of mass m_1

$$= \text{change in momentum}$$

$$= m_1 v_1 - m_1 u_1$$

$$= m_1\left[\left(\frac{m_1 - em_2}{m_1+m_2}\right)u_1 + \frac{m_2(1+e)u_2}{m_1+m_2}\right] - m_1 u_1$$

$$= \frac{m_1}{m_1+m_2}\left[(m_1 - em_2)u_1 + m_2(1+e)u_2 - (m_1+m_2)u_1\right]$$

$$= \frac{m_1}{m_1+m_2}\left[\cancel{m_1 u_1} - em_2 u_1 + m_2 u_2 + em_2 u_2 - \cancel{m_1 u_1} - m_2 u_1\right]$$

$$= \frac{m_1}{m_1+m_2}\left[-m_2 u_1(1+e) + m_2 u_2(1+e)\right]$$

$$= \frac{m_1}{m_1+m_2} \cdot m_2(1+e)(u_2 - u_1)$$

$$= \frac{m_1 m_2}{m_1+m_2}(1+e)(u_1 - u_2)$$

Similarly impulse of the blow on the sphere of mass m_2 can be found and is

$$\frac{m_1 m_2 (1+e)}{m_1+m_2}(u_1-u_2)$$

Distance at time t After the Impact.

After impact the velocity of mass m_2 relative to m_1 is

$$= v_1 - v_2 = -e(u_1-u_2)$$

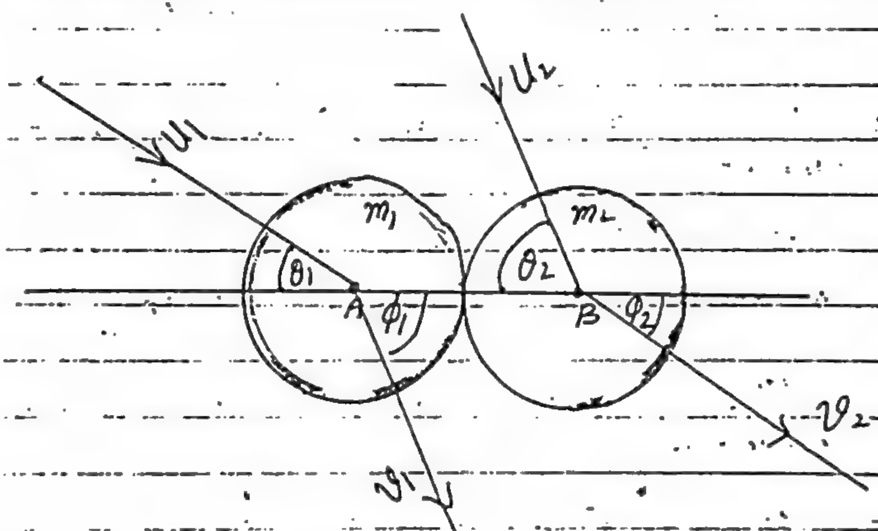
$$v_1 - v_2 = e(u_2-u_1)$$

or $v_2 - v_1 = e(u_1-u_2)$

Therefore distance between spheres after time t

$$= t(v_2-v_1)$$
$$= te(u_1-u_2)$$

Oblique Impact of Two Smooth Spheres



Suppose a smooth elastic sphere of mass m_1 impinges with a velocity u_1 obliquely on a smooth

elastic sphere of mass m_2. If the directions of motion before impact make angles θ_1, θ_2 respectively with the line joining the centres of sphere, if coefficient of (fricti) restitution is e, then find the velocities and directions of motion after impact.

Let the velocities of the spheres after impact be v_1, v_2 in the directions which are inclined at angles ϕ_1 and ϕ_2 respectively to the line of centres.

Since ther spheres are smooth, therefore there is no force perpendicular to the line joining the centres of two balls and hence velocities in that direction are unchanged. So we have

$$v_1 \sin\phi_1 = u_1 \sin\theta_1 \longrightarrow (1)$$

$$v_2 \sin\phi_2 = u_2 \sin\theta_2 \longrightarrow (2)$$

Since $u_1\cos\theta_1 - v_1\cos\phi_1$ is the normal velocity of approach $v_2\cos\phi_2 - v_1\cos\phi_1$ is the normal velocity of seperation, therefore by Laws of restitution, we have

$$\frac{v_2\cos\phi_2 - v_1\cos\phi_1}{u_1\cos\theta_1 - u_2\cos\theta_2} = e$$

$$v_2\cos\phi_2 - v_1\cos\phi_1 = e\left(u_1\cos\theta_1 - u_2\cos\theta_2\right) \rightarrow (3)$$

Since the impulsive forces acting during the collision on the two spheres along the normal at contact are equal and opposite, therefore the total momentum along this direction remain constant. So we have.

$$m_1 v_1\cos\phi_1 + m_2 v_2\cos\phi_2 = m_1 u_1\cos\theta_1 + m_2 u_2\cos\theta_2 \longrightarrow (4)$$

Multiplying (3) by m_2 and subtracting from (4), we have

$$(m_1+m_2)v_1\cos\phi_1 = (m_1 - e m_2)u_1\cos\theta_1 + m_2(1+e)u_2\cos\theta_2$$

$$v_1\cos\phi_1 = \frac{(m_1 - e m_2)u_1\cos\theta_1 + m_2(1+e)u_2\cos\theta_2}{m_1+m_2} \rightarrow (5)$$

Multiplying (3) by m_1 and adding to (4)

$$(m_1+m_2)v_2\cos\phi_2 = (1+e)m_1u_1\cos\theta_1 + (m_2 - e m_1)u_2\cos\theta_2$$

$$V_2\cos\phi_2 = \frac{(m_2 - e m_1)u_2\cos\theta_2 + (1+e)m_1u_1\cos\theta_1}{m_1+m_2} \rightarrow (6)$$

Squaring and adding (1) and (5), we get v_1^2 and hence v_1

Dividing (1) by (5), we get $\tan\phi_1$ and hence ϕ_1

Similarly squaring and adding (2) and (6) we get v_2^2 and hence v_2

Dividing (2) by (6) we get $\tan\phi_2$ and hence we get ϕ_2.

Deductions#

Result-1 # If $u_2 = 0$, then from (2).

$$v_2 \sin\phi_2 = 0$$

$$\Rightarrow \sin\phi_2 = 0 \quad \because v_2 \neq 0$$

$$\Rightarrow \phi_2 = 0$$

$\Rightarrow$ The sphere of mass m_2 will move along the line of centre. This follows independently, since the only force on m_2 is along the line of centre

Result-2 # If $m_1 = m_2$ $e = 1$, then we have from (5) and (6)

$v_1 \cos\phi_1 = u_2 \cos\theta_2$ & $v_2 \cos\phi_2 = u_1 \cos\theta_1$

Hence if two equal perfectly elastic spheres impinge, they interchange their velocities in the direction of the line of centres.

Result 3 # If $m_1 = m_2$ $e = 1$, $\theta_2 = 90^\circ$

Then

$$V_1 \cos\phi_1 = 0 \qquad \phi_1 = 90^\circ$$

i.e sphere of mass m_1 moves perpendiculars to the line of centre after collision

Also

$$V_2 \cos\phi_2 = u_1 \cos\theta_1$$

i.e the velocity of sphere moving perpendicularly before collision along normal is $u_1 \cos\theta_1$

and from (2)

$$v_2 \sin\phi_2 = u_2 \sin 90^\circ$$
$$= u_2$$

i.e velocity perpendicular to normal remain u_2.

Thus when a perpefectly elastic ball moving obliquely strikes an equal perpefectly elastic ball moving in a direction ⊥ar to the common normal, then after collision the ball moving obliquely with angle less than 90° moves perpendicularly to normal and it gives its normal compont of velocity to 2nd ball so that 2nd ball has normal velocity $u_1 \cos\theta_1$ and moves obliquely after collision.

Result 4 Impulse of the Blow #

The impulse of blow on the first sphere along normal = change produced in its momentum

$$= m_1 u_1 \cos\theta_1 - m_1 v_1 \cos\phi_1$$

$$= m_1 [u_1 \cos\theta_1 - v_1 \cos\phi_1]$$

putting $v_1 \cos\phi_1 = \dfrac{(m_1 - e m_2) u_1 \cos\theta_1 + (1+e) m_2 \cos\theta_2 \, u_2}{m_1 + m_2}$

from (5)

$$\text{Impulse} = m_1 \left[u_1 \cos\theta_1 - \frac{(m_1 - e m_2) u_1 \cos\theta_1 + (1+e) m_2 u_2 \cos\theta_2}{m_1 + m_2} \right]$$

$$= m_1 \left[\frac{u_1 \cos\theta_1 (m_1 + m_2) - m_1 u_1 \cos\theta_1 + e m_2 u_1 \cos\theta_1 - m_2 u_2 \cos\theta_2 - e m_2 u_2 \cos\theta_2}{m_1 + m_2} \right]$$

$$= \frac{m_1}{m_1 + m_2} [u_1 \cos\theta_1 \cdot m_1 + u_1 m_2 \cos\theta_1 - m_1 u_1 \cos\theta_1 + e m_2 u_1 \cos\theta_1 - m_2 u_2 \cos\theta_2 - e m_2 u_2 \cos\theta_2]$$

$$= \frac{m_1}{m_1 + m_2} [(1+e) u_1 m_2 \cos\theta_1 - u_2 \cos\theta_2 \cdot m_2 (1+e)]$$

$$= \frac{m_1 m_2}{m_1 + m_2} (1+e)(u_1 \cos\theta_1 - u_2 \cos\theta_2)$$

∴ The impulse of the blow on the other ball is equal and opposite to this.

Results **Distance at time t after impact #**

After impact the normal relative velocity of mass m_1 relative to mass $m_2 = v_2 \cos\phi_2 - v_1 \cos\phi_1$

$$= e(u_1 \cos\theta_1 - u_2 \cos\theta_2) \quad \text{by (3)}$$

therefore distance between spheres at time t

$$= t(v_2 \cos\phi_2 - v_1 \cos\phi_1)$$

$$= te(u_1 \cos\theta_1 - u_2 \cos\theta_2)$$

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Result: 6 # If $m_1 = m_2$ $e = 1$, then, we have from ⑤ and ⑥

(i) ← $V_1 \cos\varphi_1 = U_2 \cos\theta_2$ (ii) ← $V_2 \cos\varphi_2 = U_1 \cos\theta_1$

Also from ① & ② we have

(iii) ← $V_1 \sin\varphi_1 = U_1 \sin\theta_1$ & (iv) ← $V_2 \sin\varphi_2 = U_2 \sin\theta_2$

(iii) ÷ (i) ⇒

$$\tan\varphi_1 = \frac{U_1}{U_2}\tan\theta_1 \longrightarrow (V)$$

(IV) ÷ (ii)

$$\tan\varphi_2 = \frac{U_2}{U_1}\tan\theta_2 \longrightarrow (VI)$$

Multiplying (V) & (VI), we get

$$\tan\varphi_1 \tan\varphi_2 = \tan\theta_1 \tan\theta_2 \longrightarrow (VII)$$

Now if we consider line joining the centres as x-axis; then inclinations of the directions of motion befor collision are $180° - \theta_1$ & $180° - \theta_2$ and after collision these are $180° - \varphi_1$, $180° - \varphi_2$

Now

$$\tan(180° - \varphi_1) \cdot \tan(180° - \varphi_2)$$

$$= -\tan\varphi_1 \cdot -\tan\varphi_2$$

$$= \tan\varphi_1 \tan\varphi_2$$

Also $\tan(180 - \theta_1) \cdot \tan(180° - \theta_2) = \tan\theta_1 \tan\theta_2$

using these in (VII) we have

$$\tan(180 - \varphi_1) \cdot \tan(180° - \varphi_2) = \tan(180° - \theta_1)\tan(180° - \theta_2) \longrightarrow (VIII)$$

⇒ product of the slopes of the direction of motion after collision = the product of the slopes of directions of motion before collision.

Now if these equal and perfectly elastic spheres

impinge at right angles, Than

$$\tan(180-\theta_1)\cdot\tan(180^\circ-\theta_2) = -1$$

and from (viii)

$$\tan(180^\circ-\phi_1)\cdot\tan(180^\circ-\phi_2) = -1$$

$\Rightarrow$ Directions of spheres after impact are also at right angles.

Thus if two equal and perfectly elastic spheres impinge at right-angles, their directions after impact will still be at right-angles.

Loss in K.E By Impact

Two spheres of given masses moving with given velocities impinge directly. Show that there is always loss of K.E unless the elasticity is perfect.

Let m_1, m_2 be masses of the spheres and u_1, u_2 be their velocities before collision. Let v_1, v_2 be their velocities after collision.

Total K.E before impact $= \frac{1}{2}(m_1u_1^2 + m_2u_2^2)$

Total K.E after impact $= \frac{1}{2}(m_1v_1^2 + m_2v_2^2)$

Loss in K.E $= \frac{1}{2}(m_1u_1^2 + m_2u_2^2) - \frac{1}{2}(m_1v_1^2 + m_2v_2^2)$

By principle of conservation of momentum

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \longrightarrow (1)$$

By Law of restitution.

$$v_1 - v_2 = -e(u_1 - u_2) \longrightarrow (2)$$

Squaring (1) and (2)

$$(m_1v_1 + m_2v_2)^2 = (m_1u_1 + m_2u_2)^2 \longrightarrow (3)$$

$$(v_1 - v_2)^2 = e^2(u_1 - u_2)^2 \longrightarrow (4)$$

Multiplying ④ by m_1m_2 and adding to ③

$$(m_1V_1+m_2V_2)^2+m_1m_2(V_1-V_2)^2 = (m_1U_1+m_2U_2)^2+e^2m_1m_2(U_1-U_2)^2$$

$$m_1^2V_1^2+m_2^2V_2^2+m_1m_2V_1^2+m_1m_2V_2^2$$

$$=(m_1U_1+m_2U_2)^2+e^2m_1m_2(U_1-U_2)^2+m_1m_2(U_1-U_2)^2-m_1m_2(U_1-U_2)^2$$

(adding & subtracting $m_1m_2(U_1-U_2)^2$)

$$m_1(m_1V_1^2+m_2V_2^2)+m_2(m_1V_1^2+m_2V_2^2)=(m_1U_1+m_2U_2)^2+m_1m_2(U_1-U_2)^2 - m_1m_2(1-e^2)(U_1-U_2)^2$$

$$(m_1+m_2)(m_1V_1^2+m_2V_2^2)=m_1^2U_1^2+m_2^2U_2^2+m_1m_2U_1^2+m_1m_2U_2^2 - m_1m_2(1-e^2)(U_1-U_2)^2$$

$$=m_1(m_1+m_2)U_1^2+m_2(m_1+m_2)U_2^2-m_1m_2(1-e^2)(U_1-U_2)^2$$

Dividing by m_1+m_2

$$m_1V_1^2+m_2V_2^2=m_1U_1^2+m_2U_2^2-\frac{m_1m_2}{m_1+m_2}(1-e^2)(U_1-U_2)^2$$

$$m_1V_1^2+m_2V_2^2-(m_1U_1^2+m_2U_2^2)=-\frac{m_1m_2}{m_1+m_2}(1-e^2)(U_1-U_2)^2$$

$$\Rightarrow m_1U_1^2+m_2U_2^2-(m_1V_1^2+m_2V_2^2)=\frac{m_1m_2}{m_1+m_2}(1-e^2)(U_1-U_2)^2$$

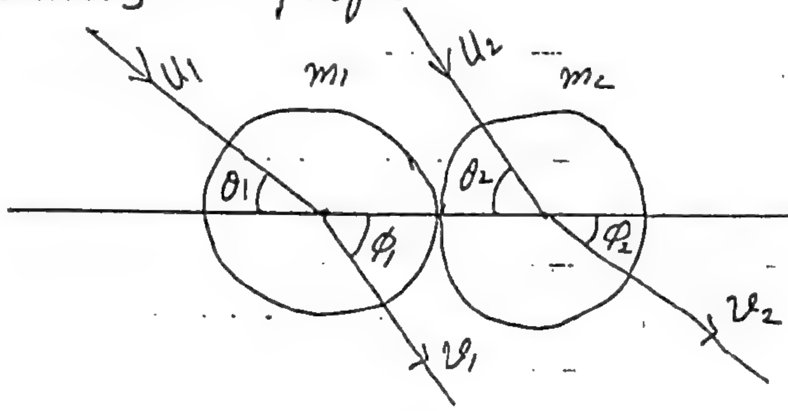
$$\Rightarrow \frac{1}{2}(m_1U_1^2+m_2U_2^2)-\frac{1}{2}(m_1V_1^2+m_2V_2^2)=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(U_1-U_2)^2$$

$$\text{Loss in K.E} = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(U_1-U_2)^2$$

Since $e<1$, this loss is +ve and there is always a loss of K.E. If $e=1$, loss in K.E is zero. Thus there is always loss in K.E unless elasticity is perfect.

Loss in K.E During Oblique Collision

When two smooth spheres impinge obliquely, the K.E is always lost by impact, unless the elasticity is perfect.



Suppose that two smooth spheres of masses m_1 & m_2 collide obliquely and their velocities are u_1, u_2 befor collision. Let θ_1, θ_2 be angles of the directions of motion before collision with line of centre. Let v_1, v_2 be the velocities after collision and φ_1, φ_2 be angles of directions with line of centre.

Since the sphere are smooth, therefore no force at right angle to line of centre and hence velocities in this direction remain same during impact.

$$v_1 \sin\varphi_1 = u_1 \sin\theta_1 \longrightarrow ①$$

$$v_2 \sin\varphi_2 = u_2 \sin\theta_2 \longrightarrow ②$$

N $\quad u_1^2 = (u_1)_n^2 + (u_1)_t^2 = u_1^2\cos^2\theta_1 + u_1^2\sin^2\theta_1$

$$u_2^2 = (u_2)_n^2 + (u_2)_t^2 = u_2^2\cos^2\theta_2 + u_2^2\sin^2\theta_2$$

$$v_1^2 = (v_1)_n^2 + (v_2)_t^2 = v_1^2\cos^2\varphi_1 + v_1^2\sin^2\varphi_1$$

$$v_2^2 = (v_2)_n^2 + (v_2)_t^2 = v_2^2\cos^2\varphi_2 + v_2^2\sin^2\varphi_2$$

$$\text{Loss in K:E} = \frac{1}{2}(m_1u_1^2 + m_2u_2^2) - \frac{1}{2}(m_1v_1^2 + m_2v_2^2)$$

Loss in K.E $= \frac{1}{2}\left[m_1(u_1^2\cos^2\theta_1 + u_1^2\sin^2\theta_1) + m_2(u_2^2\cos^2\theta_2 + u_2^2\sin^2\theta_2)\right]$

$- \frac{1}{2}\left[m_1(v_1^2\cos^2\varphi_1 + v_1^2\sin^2\varphi_1) + m_2(v_2^2\cos^2\varphi_2 + v_2^2\sin^2\varphi_2)\right]$

Using $v_1\sin\varphi_1 = u_1\sin\theta_1$ $\quad v_2\sin\varphi_2 = u_2\sin\theta_2$

from ① & ②

Loss in K.E

$= \frac{1}{2}\left[m_1(u_1^2\cos^2\theta_1 + u_1^2\sin^2\theta_1) + m_2(u_2^2\cos^2\theta_2 + u_2^2\sin^2\theta_2)\right]$

$- \frac{1}{2}\left[m_1(v_1^2\cos^2\varphi_1 + u_1^2\sin^2\theta_1) + m_2(v_2^2\cos^2\varphi_2 + u_2^2\sin^2\theta_2)\right]$

$= \frac{1}{2}m_1u_1^2\cos^2\theta_1 + \frac{1}{2}m_2u_2^2\cos^2\theta_2 - \frac{1}{2}m_1v_1^2\cos^2\varphi_1 - \frac{1}{2}m_2v_2^2\cos^2\varphi_2$

It follows that it is only the velocity components along the line of centres that can affect a change in K.E of the spheres.

Now by principle of conservation of momentum.

$m_1v_1\cos\varphi_1 + m_2v_2\cos\varphi_2 = m_1u_1\cos\theta_1 + m_2u_2\cos\theta_2 \rightarrow ③$

By Law of restitution.

$v_1\cos\phi_1 - v_2\cos\phi_2 = -e(u_1\cos\theta_1 - u_2\cos\theta_2) \rightarrow ④$

Squaring ③ & ④, we have.

$(m_1v_1\cos\varphi_1 + m_2v_2\cos\varphi_2)^2 = (m_1u_1\cos\theta_1 + m_2u_2\cos\theta_2)^2 \rightarrow ⑤$

$(v_1\cos\varphi_1 - v_2\cos\varphi_2)^2 = e^2(u_1\cos\theta_1 - u_2\cos\theta_2)^2 \rightarrow ⑥$

Multiplying ⑥ by m_1m_2 and adding to ⑤

$(m_1v_1\cos\varphi_1 + m_2v_2\cos\varphi_2)^2 + m_1m_2(v_1\cos\varphi_1 - v_2\cos\varphi_2)^2$

$= (m_1u_1\cos\theta_1 + m_2u_2\cos\theta_2)^2 + m_1m_2e^2(u_1\cos\theta_1 - u_2\cos\theta_2)^2$

$$m_1^2 V_1^2 \cos^2\varphi_1 + m_2^2 V_2^2 \cos^2\varphi_2 + m_1 m_2 V_1^2 \cos^2\varphi_1 + m_1 m_2 V_2^2 \cos^2\varphi_2$$

$$= (m_1 u_1 \cos\theta_1 + m_2 u_2 \cos\theta_2)^2 + m_1 m_2 e^2 (u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

$$+ m_1 m_2 (u_1\cos\theta_1 - u_2\cos\theta_2)^2 - m_1 m_2 (u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

$$m_1(m_1 V_1^2 \cos^2\varphi_1 + m_2 V_2^2 \cos^2\varphi_2) + m_2(m_1 V_1^2\cos^2\varphi_1 + m_2 V_2^2 \cos^2\varphi_2)$$

$$\Rightarrow m_1^2 u_1^2 \cos^2\theta_1 + m_2^2 u_2^2 \cos^2\theta_2 + m_1 m_2 u_1^2 \cos^2\theta_1 + m_1 m_2 u_2^2 \cos^2\theta_2$$

$$+ m_1 m_2 (1-e^2)(u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

$$\Rightarrow (m_1+m_2)(m_1 V_1^2 \cos^2\varphi_1 + m_2 V_2^2 \cos^2\varphi_2)$$

$$= m_1(u_1^2\cos^2\theta_1 \cdot m_1 + m_2 u_2^2 \cos^2\theta_2) + m_2(m_1 u_1^2 \cos^2\theta_1 + m_2 u_2^2 \cos^2\theta_2)$$

$$- m_1 m_2 (1-e^2)(u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

$$(m_1+m_2)(m_1 V_1^2 \cos^2\varphi_1 + m_2 V_2^2 \cos^2\varphi_2) = (m_1+m_2)(m_1 u_1^2 \cos^2\theta_1 + m_2 u_2^2 \cos^2\theta_2)$$

$$- m_1 m_2 (1-e^2)(u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

Dividing by m_1+m_2

$$m_1 V_1^2 \cos^2\varphi_1 + m_2 V_2^2 \cos^2\varphi_2 = m_1 u_1^2 \cos^2\theta_1 + m_2 u_2^2 \cos^2\theta_2$$

$$- \frac{m_1 m_2 (1-e^2)}{m_1+m_2}(u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

$$m_1 u_1^2 \cos^2\theta_1 + m_2 u_2^2 \cos^2\theta_2 - (m_1 V_1^2 \cos^2\varphi_1 + m_2 V_2^2 \cos^2\varphi_2)$$

$$= \frac{m_1 m_2}{m_1+m_2}(1-e^2)(u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

$$\Rightarrow \frac{1}{2}(m_1 u_1^2 \cos^2\theta_1 + m_2 u_2^2 \cos^2\theta_2) - \frac{1}{2}(m_1 V_1^2 \cos^2\varphi_1 + m_2 V_2^2 \cos^2\varphi_2)$$

$$= \frac{m_1 m_2}{2(m_1+m_2)}(1-e^2)(u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

$$\text{Loss in K.E} = \frac{m_1 m_2}{2(m_1+m_2)}(1-e^2)(u_1\cos\theta_1 - u_2\cos\theta_2)^2$$

Hence we see that in any impact (direct or oblique), unless the co-efficient of restitution be unity, some K.E is lost or is transformed.

This missing K.E is converted into molecular energy and chiefly reappears in the shape of heat.

Note # If $u_2 = 0$ and $m_2 \to \infty$, then loss in K.E is

Loss in K.E in case of oblique collision

$$= \lim_{m_2 \to \infty} \frac{m_1}{2\left(\frac{m_1}{m_2}+1\right)} (1-e^2)(u_1^2\cos^2\theta_1 + 0)$$

$$= \frac{1}{2} m_1 (1-e^2)(u_1^2 \cos^2\theta_1)$$

and loss in K.E in case of direct collision

$$= \lim_{m_2 \to \infty} \frac{m_1}{2\left(\frac{m_1}{m_2}+1\right)} (1-e^2)(u_1 - 0)^2$$

$$= \frac{1}{2} m_1 (1-e^2) u_1^2$$

which same as the loss of K.E of a sphere colliding obliquely and directly with a fixed plane.

Problems About Collision of a sphere with fixed Horizontal or Vertical Plane

Problem # 1 Determine the co-efficient of restitution for a steel ball dropped from rest at a height h_1 above a heavy horizontal steel plate if the height at the 2nd rebound is h_2.

Sol # Motion Before 1st Impact #

$\because$ ball is dropped from rest

$\therefore$ Initial velocity $= v_i = 0$

Distance covered $= h$

acc $= g$

Suppose final velocity = velocity of impact $= V_f = U = ?$

$$2as = V_f^2 - V_i^2$$

$$2gh = u^2 - 0$$

$$u = \sqrt{2gh}$$

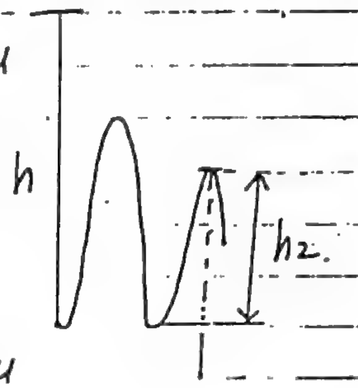
$\Rightarrow$ Velocity at 1st impact $= u = \sqrt{2gh}$

Motion after First Impact

$\because$ ball collides directly

$\therefore$ Velocity after 1st rebound $= eu$

This is also the velocity of ball before 2nd impact.



Motion After 2nd Impact

Velocity before impact $= eu$

Velocity after impact $= e \cdot eu = e^2u$

Distance covered $= h_2$

acc $= -g$

final velocity at height $h_2 = V_f = 0$

$$2as = V_f^2 - V_i^2$$

$$-2gh_2 = 0 - (e^2u)^2$$

$$-2gh_2 = -e^4u^2$$

putting $u = \sqrt{2gh}$

$$2gh_2 = +e^4(\sqrt{2gh})^2$$

$$\cancel{2g}h_2 = e^4 \cdot (\cancel{2g}h)$$

$$e = \left(\frac{h_2}{h}\right)^{\frac{1}{4}}$$

Problem#2# A ball is released from rest and drops a distance h onto the horizontal surface of heavy steel plate. If the ball rebounds to height h', determine the co-efficient of restitution e. What is the fraction n of the original energy which is lost.

Sol# **Motion Before Impact#**

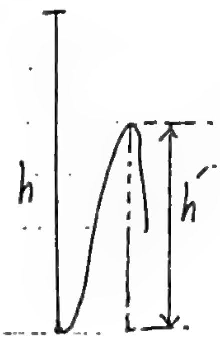
$\because$ ball is dropped from rest

$\therefore$ Initial vel $= v_i = 0$

Distance $=$ height $= h$

acc $= g$

Velocity of impact $=$ Final velocity on striking the steel plate $= V_f = u = ?$



$2as = V_f^2 - V_i^2$

$2gh = u^2 - 0$

$\Rightarrow u = \sqrt{2gh}$

Motion After Impact#

Velocity of impact $= u = \sqrt{2gh}$

Velocity after impact $= eu$

acc $= -g$

Distance $=$ height reached $= h'$

Final velocity at height reached $= 0 = V_f$

$\therefore$ $2as = V_f^2 - V_i^2$

$-2gh' = 0 - (eu)^2$

$2gh' = e^2 u^2$

using $u = \sqrt{2gh}$

$2gh' = e^2 (\sqrt{2gh})^2$

$2gh' = e^2 \cdot 2gh$

$e^2 = \frac{h'}{h} \Rightarrow e = \sqrt{\frac{h'}{h}}$

Energy Lost #

$$K.E \text{ before impact} = \frac{1}{2}mu^2$$

$$= \frac{1}{2}m(\sqrt{2gh})^2$$

$$= mgh$$

$$K.E \text{ after impact} = \frac{1}{2}m(eu)^2$$

$$= \frac{1}{2}m e^2 u^2$$

$$= \frac{1}{2}m\left(\sqrt{\frac{h'}{h}}\right)^2 \cdot 2gh$$

$$= \frac{1}{\cancel{2}}m \cdot \frac{h'}{\cancel{h}} \cdot \cancel{2}g\cancel{h}$$

$$= mgh'$$

$$K.E. \text{ lost} = mgh - mgh' = mg(h-h')$$

$$\text{fraction } n \text{ of energy lost} = \frac{K.E \text{ lost}}{\text{original energy}}$$

$$= \frac{mg(h-h')}{mgh}$$

$$= \frac{h-h'}{h}$$

Problem #3# A heavy elastic ball is dropped upon a smooth horizontal floor from a height of 40 ft. and after rebounding thrice it is observed to attain a height of 20 ft. Find the co-efficient of restitution and total energy lost after three impact.

Sol #   Downward Motion before 1st Impact #

$$\text{Initial velocity} = v_i = 0$$

$$\text{Final velocity} = \text{Impact velocity} = V_f = u = ?$$

$$acc = g$$

$$\text{Distance} = \text{height} = 40 ft$$

$$2as = V_f^2 - V_i^2$$
$$2g \cdot 40 = u^2 - 0$$
$$2 \times 32 \cdot 40 = u^2$$
$$u = \sqrt{2 \times 32 \times 40} = \sqrt{2 \times 8 \times 4 \times 8 \times 5}$$
$$= 8 \times 2\sqrt{10} = 16\sqrt{2}.$$

Velocity of rebound after 1st impact $= eu$

" " " 2nd impact $= e^2 u$

" " " 3rd impact $= e^3 u$

Motion after 3rd Rebound

Velocity after 3rd rebound $= e^3 u$

Distance $=$ height $= 20$ ft

acc $= -g$

Final velocity at height 20ft $= 0$

$$2as = V_f^2 - V_i^2$$
$$-2g \cdot 20 = 0 - (e^3 u)^2$$
$$2 \cdot 32 \times 20 = e^6 (16\sqrt{10})^2 \qquad \because u = 16\sqrt{10}$$
$$2 \times 32 \times 20 = 256 \times 10\, e^6$$
$$e^6 = \frac{2 \times \cancel{32}^{2} \times \cancel{20}^{2}}{\cancel{256}_{16} \times \cancel{10}} = \frac{1}{2}$$
$$e = \left(\frac{1}{2}\right)^{\frac{1}{6}}$$

K.E Lost

K.E before 1st impact $= \frac{1}{2} m u^2$

K.E after 3rd impact $= \frac{1}{2} m e^6 u^2$

K.E Lost $= \frac{1}{2} m u^2 - \frac{1}{2} m e^6 u^2$

$$= \frac{1}{2} m u^2 (1 - e^6)$$

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Problem#4# A heavy elastic ball is dropped from rest upon a horizontal floor from some height. After rebounding twice, it attains a height of 10 ft after 2nd rebound. If $e = (\frac{1}{2})^{\frac{1}{4}}$, find height through which it is dropped.

Sol# Let h be height through which it is dropped.

Downward Motion Before 1st Impact#

Initial velocity $= V_i = 0$

acc $= g$

Distance $= h = ?$

Final velocity $= V_f = u =$ Velocity of 1st impact $= ?$

$$2as = V_f^2 - V_i^2$$

$$2gh = u^2 - 0$$

$$u = \sqrt{2gh}$$

Velocity after 1st rebound $= eu$

Motion after 2nd Rebound#

Velocity after impact $= e^2u$

Height reached $= 10\,ft = s$

acc $= -g$

Final velocity $= V_f = 0$

$$2as = V_f^2 - V_i^2$$

$$-2g(10) = 0 - (e^2u)^2$$

$$20g = e^4u^2$$

putting $u = \sqrt{2gh}$

$$20g = e^4(\sqrt{2gh})^2$$

$$20g = e^4 \cdot 2gh$$

$$h = \frac{10}{e^4} = \frac{10}{((\frac{1}{2})^{1/4})^4} = \frac{10}{1/2} = 20\,ft$$

So ball is dropped from a height of 20 ft.

Problem#5# A ball is dropped on the floor from a height h. If the co-efficient of restitution is e, find the height of the ball at the nth rebound#

Sol# **Downward Motion from height h#**

Initial Velocity $= v_i = 0$

acc $= g$

Distance $= h$

Final velocity $= V_f = u =$ Velocity of impact $= ?$

$$2as = V_f^2 - V_i^2$$

$$\Rightarrow 2gh = u^2$$

$$u = \sqrt{2gh}$$

Velocity after 1st rebound $= eu$

Velocity after 2nd rebound $= e^2u$

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$\therefore$ Velocity after nth rebound $= e^n u$

Motion after nth Rebound#

$\because$ Height reached $= h_1 = ?$

Initial velocity $= v_i = e^n u$

Final velocity at height reached $= V_f = 0$

acc $= -g$

$$2as = V_f^2 - V_i^2$$

$$-2gh_1 = 0 - (e^n u)^2$$

$$2gh_1 = e^{2n} u^2 = e^{2n}(2gh)$$

$$\Rightarrow h_1 = e^{2n} h$$

$\therefore$ Height reached $= e^{2n} h$.

Problem# A small sphere is dropped from a height of $1.2m$ on a smooth horizontal plane rebounds to a height of $1m$. Find co-efficient of

restitution

Sol# <u>Downward Motion Befor Impact #</u>

Distance covered = height fallen = 1.2 m h

acc = g

Since the sphere is dropped, Therefore

Initial Velocity = $V_f = v$ = Velocity of impact

$$V_f^2 - v_i^2 = 2as$$

$$V^2 - 0 = 2g(1.2)$$

$$V^2 = 2gh$$

$$V = \sqrt{2gh}$$

Velocity before impact = $V = \sqrt{2gh}$

Velocity after 1st impact = ev

<u>Motion After 1st Rebound #</u>

Initial velocity = Velocity after rebound = ev

Distance = height reached = 1m

Final velocity at height reached = 0 m/sec

acc = $-g$

$$V_f^2 - V_i^2 = 2as$$

$$0 - (ev)^2 = -2g(-1)$$

$$e^2v^2 = 2g$$

putting $v = \sqrt{2gh}$

$$e^2(\cancel{2}g h) = \cancel{2}\cancel{g}$$

$$e^2 = \frac{1}{h} = \frac{1}{1.2} = \frac{10}{12} = \frac{5}{6}$$

$$e = \sqrt{\frac{5}{6}}$$

Problem # 6 (2001) A heavy ball drops force ceiling of room and after rebounding twice from the floor reaches a height equal to one half of the

ceiling. Show that, the co-efficient of restitution is equal to $(\frac{1}{2})^{\frac{1}{4}}$

Sol # Let the height of ceiling be h

Motion Befor 1st Impact #

Distance = heigh fallen = h

acc = g

final velocity = $V_f = u = ?$

Initial velocity = $v_i = 0$ because the ball is dropped

$2as = V_f^2 - V_i^2$

$2gh = u^2 - 0$

$u = \sqrt{2gh}$

Velocity after 1st impact = eu

velocity after 2nd impact = e^2u.

Motion After 2nd Impact #

Initial velocity = e^2u.

acc = $-g$

Distance = height reached = $\frac{h}{2}$

final velocity = $V_f = 0$

$2as = V_f^2 - V_i^2$

$-2g\frac{h}{2} = 0 - (e^2u)^2$

$gh = e^4 \cdot 2gh$

$e^4 = \frac{1}{2}$

$e = (\frac{1}{2})^{\frac{1}{4}}$

Problem # A rubber ball drops from a height h and after rebounding twice from the ground, it reaches a height of $h/4$. Find the Co-efficient

of restitution. What would be the co-efficient of restitution had the ball reached a height $\frac{h}{4}$ after rebounding three times.

Sol# Let e be the co-efficient of restitution.

Downward Motion from Height h#

Initial velocity $= v_i = 0$

Distance $= h$

Final velocity $= V_f = v =$ Impact velocity $= ?$

$acc = g$

$$v^2 - v_i^2 = 2gh$$

$$v^2 - 0 = 2gh$$

$$v = \sqrt{2gh}$$

Velocity after 1st impact $= ev$

Motion after 2nd Rebound#

Velocity after impact $= ev$

Velocity of rebound $= e \cdot ev = e^2 v = v_i$

Height reached $= h/4$

$acc = -g$

Final velocity at height reached $= V_f = 0$

$$2as = V_f^2 - V_i^2$$

$$-2g \cdot \frac{h}{4} = 0 - (e^2 v)^2$$

$$\frac{gh}{2} = e^4 v^2$$

putting $v = \sqrt{2gh}$

$$\frac{gh}{2} = e^4 \cdot 2gh$$

$$e^4 = \frac{1}{4}$$

$$e = \left(\frac{1}{4}\right)^{\frac{1}{4}}$$

Problem#7# A small sphere is dropped onto a horizontal plane from a height 'h'. If the co-efficient of restitution between the sphere and plane is e, find in terms of h and e, the height to which the particle rises after the 1st, 2nd and third rebound, showing that these heights are in geometric progression deduce the total distance travelled by the sphere after n rebounds and time taken to cover the distance after n rebounds if the elasticity is imperfect. Also find the total distance travelled by the sphere before it comes to rest and the time that elapses.

Sol# **Downward Motion before 1st Impact#**

Initial velocity $= v_i = 0$

Height $= h$

acc $= g$

Final velocity $=$ velocity of impact $= v_f = u$

$$2as = v_f^2 - v_i^2$$

$$2gh = u^2 - 0$$

$$u = \sqrt{2gh}$$

Time taken $= t = ?$

$$v_f = v_i + at$$

$$\sqrt{2gh} = 0 + gt$$

$$t_1 = \frac{\sqrt{2gh}}{g} = \sqrt{\frac{2h}{g}}$$

Motion after 1st Rebound#

Velocity after 1st rebound $= eu$

acc $= -g$

Let height reached $= h_1$

final velocity $= 0$

$$2as = v_f^2 - v_i^2$$

$$-2gh_1 = 0 - e^2u^2 \Rightarrow \boxed{h_1 = e^2 h}$$

Time taken $= t_1 = ?$

$$V_f = V_i + at$$

$$0 = eu - gt_1$$

$$t_1 = \frac{eu}{g}$$

$$\boxed{t_1 = et}$$

It will reach the horizontal plane again with the same velocity eu, in the same time $t_1 = et$ after moving the same distance e^2h

Thus after 1st rebound & before 2nd rebound total distance $= 2e^2h$

and time taken $= 2et$

Height reached after 2nd rebound $= e^4h$

time taken $= e^2t$

Distance covered after 2nd rebound and before 3rd rebound $= 2e^4h = 2e^{2(2)}h$

time taken $= 2e^2t = 2e^2t$

- - - - - - - - - - - - - - - - - -

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Distance covered after nth rebound $= 2e^{2n}h$

Time taken during this distance $= 2e^nt$

Distance covered during n rebounds

$$= h + 2e^2h + 2e^4h + \cdots + 2e^{2n}h$$

$$= h + 2h(e^2 + e^4 + \cdots + e^{2n})$$

$$= h + 2h\left\{\frac{e^2(1-(e^2)^n)}{1-e^2}\right\}$$

$$= h + 2h\left\{\frac{e^2(1-e^{2n})}{1-e^2}\right\}$$

$$= h\left[1 + \frac{2e^2(1-e^{2n})}{1-e^2}\right]$$

$$= h\left[\frac{1-e^2+2e^2(1-e^{2n})}{1-e^2}\right]$$

$$= \frac{h\left[1+e^2-2e^{2n+2}\right]}{1-e^2}$$

Total time taken after n rebounds.

$$= t+2et+2e^2t+ \cdots \cdots +2e^nt$$

$$= t+2t(e+e^2+e^3+ \cdots + e^n)$$

$$= t+2t.\frac{e(1-e^n)}{1-e}$$

$$= t\left[1+\frac{2e(1-e^n)}{1-e}\right] = \frac{t\left[1-e+2e-2e^{n+1}\right]}{1-e}$$

$$= t\left[\frac{1+e-2e^{n+1}}{1-e}\right]$$

Total Distance Covered before the sphere comes to rest

$$= h+2e^2h+ \cdots +2e^4h+2e^6h+ \cdots$$

$$= h+2h(e^2+e^4+e^6+ \cdots)$$

$$= h+2h \cdot \frac{e^2}{1-e^2}$$

$$= h\left[1+\frac{2e^2}{1-e^2}\right] = h\left[\frac{1+e^2}{1-e^2}\right]$$

Total time taken before coming to rest

$$= t+2et+2e^2t+2e^3t+ \cdots$$

$$= t+2t(e+e^2+e^3+ \cdots)$$

$= t + 2t(e + e^2 + e^3 + \cdots\cdots\cdots)$

$= t\left[1 + \frac{2e}{1-e}\right] = t\left[\frac{1+e}{1-e}\right]$

$= \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)$

Problem # 8 #(2007) A particle falls from a height h in time t, upon fixed horizontal. It rebounds and reaches a max. height h', in time t'. Show that

$t' = et$ & $h' = e^2 h$

Sol # Downward Motion #

Let the particle acquire velocity v in time t

Initial velocity $= v_i = 0$

Distance $= h$

acc $= g$

Final velocity $= v$

$2as = V_f^2 - V_i^2$

$2gh = V^2 - 0$

$V = \sqrt{2gh}$

time taken

$V_f = v_i + at$

$\sqrt{2gh} = 0 + gt$

$t = \frac{\sqrt{2gh}}{g} = \sqrt{\frac{2h}{g}}$

Motion after Rebound #

Velocity after rebound $= ev$

acc $= -g$

time $= t'$

time $= t'$

Distance $= h'$

Final velocity at height $h' = V_f = 0$

$$2as = V_f^2 - V_i^2$$

$$-2gh' = 0 - e^2 v^2$$

$$h' = e^2 h$$

Time taken $= t' = ?$

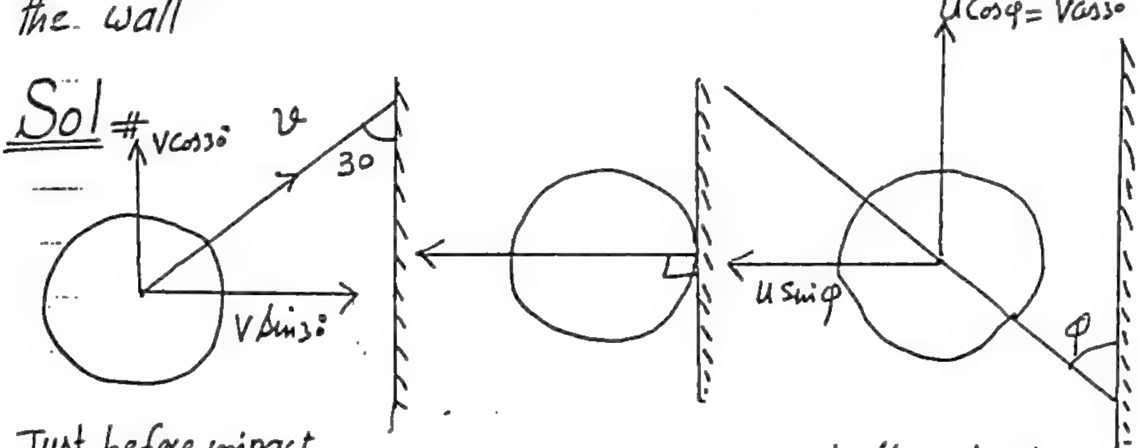
$$V_f = V_i + at$$

$$0 = ev - gt'$$

$$t' = e\frac{v}{g} = et \qquad \boxed{t' = et}$$

Problem # 9 # A smooth sphere is free to move on a horizontal surface. It is projected towards a vertical wall with speed v at an angle of $30°$ to the wall. If the co-efficient of restitution between the sphere and wall is $1/3$, find the velocity of the sphere after it hits the wall

Sol #



Just before impact — Just after impact

Since there is no impulse parallel to the wall, the velocity components in this direction are not changed by the impact.

Let U be velocity after impact and is inclined at angle φ with wall. Then.

$$U\cos\varphi = V\cos 30° = \frac{\sqrt{3}}{2}V \longrightarrow (1)$$

By Law of restitution taking +ve velocities towards R.H.S

$$\frac{-U\sin\varphi - 0}{V\sin 30^\circ - 0} = -\frac{1}{3}$$

$$U\sin\varphi = \frac{1}{3}V\sin 30^\circ$$

$$U\sin\varphi = \frac{1}{3}V.\frac{1}{2} = \frac{V}{6} \longrightarrow ②$$

Squaring and adding ① & ②

$$U^2 = \frac{3}{4}V^2 + \frac{V^2}{36} = \left(\frac{27+1}{36}\right)V^2$$

$$= \frac{28}{36}V^2$$

$$U = \sqrt{\frac{28}{36}}\,V = \frac{2\sqrt{7}}{6}v$$

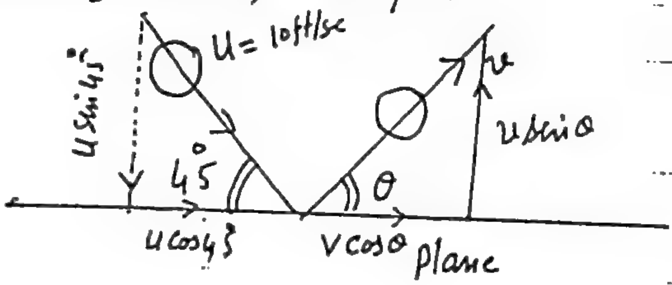
Dividing ② by ①

$$\tan\varphi = \frac{\sqrt{3}}{9}$$

$$\varphi = \tan^{-1}\left(\frac{\sqrt{3}}{9}\right) \text{ to the wall.}$$

Problem#10# A smooth ball moving with a velocity of 10ft/sec impinges on a smooth fixed plane at angle of 45°, if the co-efficient of restitution be 4/5, find the velocity and direction of motion of wall after impact.

Sol# Let u be velocity before impact and v be velocity of ball after impact..



Since the ball and plane are smooth, therefore velocity components parallel to wall (common tangent) remain same. Let θ be inclination of v with plane. Then

$$V\cos\theta = u\cos 45^\circ$$

$$V\cos\theta = 10\cdot\frac{1}{\sqrt{2}} = 5\sqrt{2} \longrightarrow (1)$$

By Law of restitution by taking velocity +ve upward $\perp$ar to plane.

$$-\frac{u\sin 45^\circ - 0}{u\sin\theta - 0} = -\frac{5}{4} = -5/4$$

$$u\sin 45^\circ = \frac{5}{4}V\sin\theta$$

$$V\sin\theta = \frac{4}{5}u\sin 45^\circ$$

$$= 4/5\cdot 10\cdot\frac{1}{\sqrt{2}} = 4\sqrt{2} \longrightarrow (2)$$

By squaring and adding (1) & (2)

$$v^2 = 50 + 32 = 82$$

$$v = \sqrt{82} = 9.06 \text{ ft/sec}$$

Dividing (2) by (1)

$$\tan\theta = 4/5$$

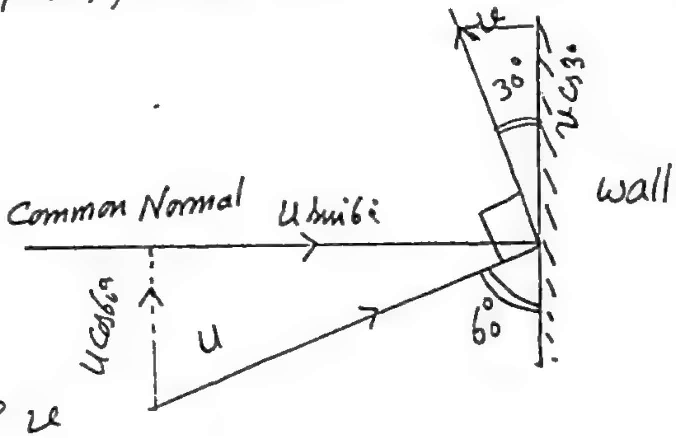
$$\theta = \tan^{-1}(4/5) = 38^\circ 40' \text{ with plane.}$$

By Muhammad Hussain Lecturer (Math) Asghar Mall College.

Problem # 11 # A smooth sphere travelling on a horizontal ground impinges obliquely on a vertical wall and rebounds at right angle to its original direction of motion. If the sphere is moving at 60° to the wall before impact, find the value of e $(e = \frac{1}{3})$

Sol #

Let u be the velocity before impact and v be velocity after impact.



The direction of v makes angle 30° with wall

$$V\cos 30^\circ = u\cos 60^\circ \longrightarrow (1)$$

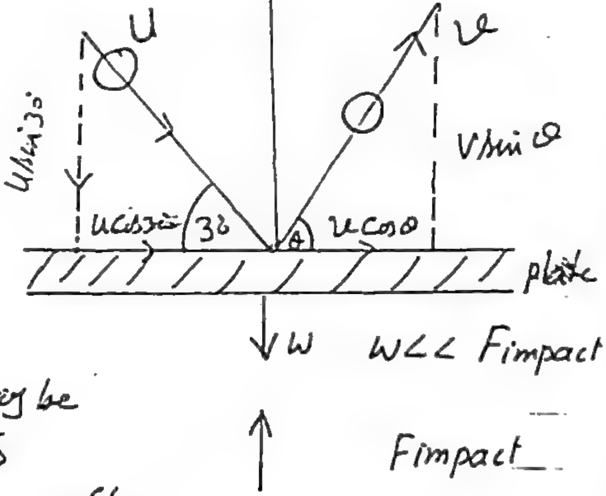
$$V\sin 30^\circ = e\,u\sin 60^\circ \longrightarrow (2)$$

Dividing ① by ②

$$\cot 30^\circ = \frac{1}{e}\cot 60^\circ$$

$$e = \frac{\cot 60^\circ}{\cot 30^\circ} = \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}} = \frac{1}{3}$$

Problem #12 # A smooth ball is projected onto the heavy smooth plate with a velocity of 50ft/sec at an angle of 30° with plate. If the effective co-efficient of restitution is 0.5. Compute the rebound velocity and direction of rebound



Sol # Let $u = 50$ ft/sc be velocity of impact and v be velocity of rebound with angle θ with plate.

The mass of plate may be considered infinite and its corresponding velocity zero after impact

Since the surfaces are smooth, therefor there is no change in velocity in common tangent (along plate) direction, we have

$$v\cos\theta = u\cos 30^\circ$$

$$= 50\cdot\frac{\sqrt{3}}{2} = 25\sqrt{3} \longrightarrow ①$$

Also by Law of restitution.

$$v\sin\theta = e\,u\sin 30^\circ$$

$$= .5\times 50\times\frac{1}{2} = 12.5 \longrightarrow ②$$

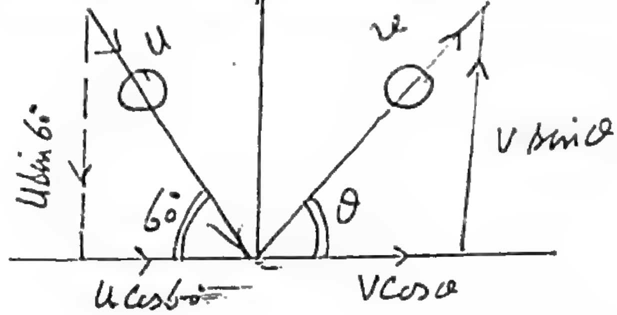
Squaring and adding ① & ②

$$v^2 = 125\times 3 + (12.5)^2$$

$$v = 45.1 \text{ ft/sec}$$

$$\tan\theta = \frac{12.5}{25\sqrt{3}} \Rightarrow \theta = \tan^{-1}\left(\frac{12.5}{25\sqrt{3}}\right) = 16.10^\circ \text{ Ans.}$$

Problem # 13 # A particle moving with a speed of 30 ft/sec in a direction making an angle of $60°$ with horizontal strikes a smooth horizontal plane and rebounds, the co-efficient of restitution being $1/3$. Find the speed and direction of motion of the particle immediately after impact



Sol # Let $u = 30$ ft/sec be velocity of projection and v be the velocity of rebound.

Along the tangential direction velocity components remain same

$$u \cos 60° = v \cos\theta$$

$$v\cos\theta = 30 \times \frac{1}{2} = 15 \quad \longrightarrow (1)$$

By Law of restitution.

$$v \sin\theta = e\, u \sin 60°$$

$$= \frac{1}{3} \cdot 30 \cdot \frac{\sqrt{3}}{2}$$

$$= 5\sqrt{3} \quad \longrightarrow (2)$$

squaring and adding (1) & (2)

$$v^2 = 225 + 75$$

$$= 300$$

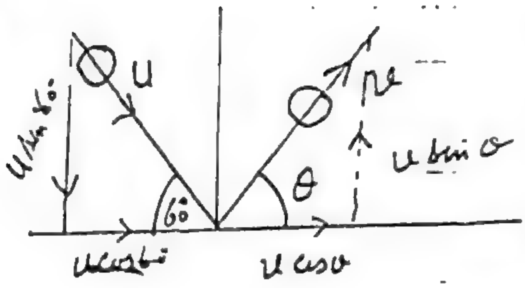
$$v = \sqrt{300} = 10\sqrt{3}$$

$$= 17.32 \text{ ft/sec}.$$

Dividing (2) by (1) $\quad \tan\theta = \frac{5\sqrt{3}}{15} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 60°$$

Problem # 14 # A ball weighing one pound and moving with a velocity of 8 ft/sec, impinges a smooth fixed plane in a direction making angle $60°$ with the plane. Find the velocity and direction after impact; the co-efficient of restitution being $1/2$. Find also the loss in K·E and the impulse in the plane due to impact.



Sol# Let $u = 8$ ft/sec be the velocity of impact of the ball. Let v be velocity of ball after impact making angle θ with plane.

Along tangential direction, there is no change in vel.

$$v\cos\theta = u\cos 60^\circ$$
$$= 8\cdot\frac{1}{2} = 4 \longrightarrow (1)$$

By Law of restitution.

$$v\sin\theta = e u \sin 60^\circ$$
$$= \frac{1}{2}\,8\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3} \longrightarrow (2)$$

squaring and adding (1) & (2)

$$v^2 = 16 + 12 = 28$$
$$v = \sqrt{28} = 5.29 \text{ ft/sec.}$$

Dividing (2) by (1)

$$\tan\theta = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$$
$$\theta = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) = 49^\circ.6'$$

In FPS-system unit of mass is slug and unit of weight is pound.

$$W = mg$$
$$1 = m\cdot 32 \Rightarrow m = \frac{1}{32}\text{ slug}$$

$$\text{Loss in K.E} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= \frac{1}{2}m(u^2 - v^2) = \frac{1}{2}\cdot\frac{1}{32}(64 - 48)$$
$$= \frac{1}{64}(36) = \frac{18}{32} = \frac{9}{16}\text{ ft-poundals}$$

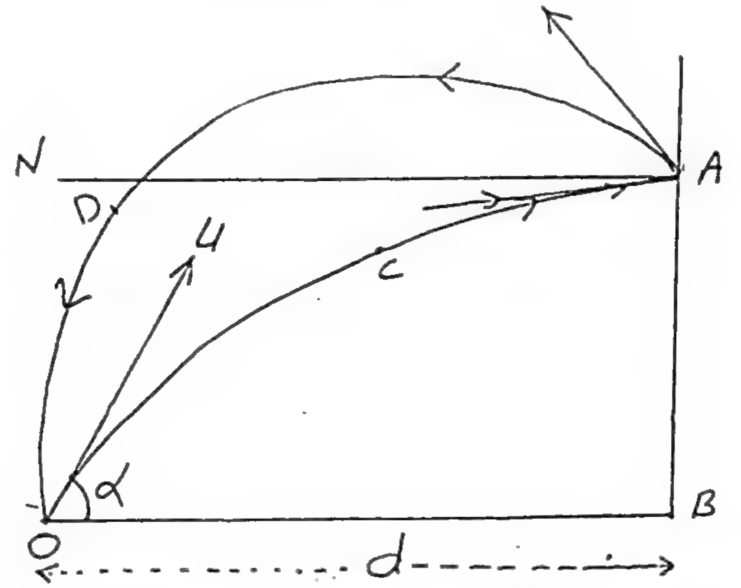
$$\text{Impulse} = \text{change in momentum along Common Normal}$$
$$= -mu\sin 60^\circ - mv\sin\theta$$
$$= -mu\sin 60^\circ - m e u\sin 60^\circ \quad \text{using (2)}$$
$$= -m(1+e)u\sin 60^\circ$$
$$= -1\times(1+1/2)\,8\times\frac{\sqrt{3}}{2} = -6\sqrt{3} = -10.4\text{ lb ft-sec}$$

Impulse $= 10.4$ lb-ft-sec.

Problem# 15# A ball is projected with a velocity u at an angle α with horizontal, from a point distant d, from a smooth vertical wall in a plane perpendicular to it. After rebounding from the wall it returns to the point of projection. Prove that

$$u^2 \sin 2\alpha = gd\left(1+\frac{1}{e}\right)$$

Sol #



Let O be the point of projection, AB be the wall and AN ⊥ar to it. Let t_1 be the time for motion from O to A along the path OCA and t_2 be the time for return along path ADO.

$d =$ $t_1 \times$ (Horizontal Component of velocity before striking the wall).

$$d = t_1 u\cos\alpha \longrightarrow ①$$

Now vertical component of velocity is not effected due to impact and horizontal component (Normal) becomes $e u\cos\alpha$ after impact. Therefore

$d =$ $t_2 \times$ (horizontal Component of velocity after impact with wall)

$$= t_2 e u\cos\alpha \longrightarrow ②.$$

$\Rightarrow \frac{d}{e} = t_2 u \cos\alpha \longrightarrow (2)$

Adding (1) & (2)

$(t_1+t_2) u \cos\alpha = d + \frac{d}{e} = d(1+\frac{1}{e}) \longrightarrow (3)$

$\because$ The wall is smooth, the vertical component of velocity $u\sin\alpha$ is unchanged and when the particle returns O its vertical velocity is $u\sin\alpha$.

Vertical motion#

Initial vertical velocity $= u\sin\alpha$.

Distance covered upt point $O = 0$

$acc = -g$

time $= t_1+t_2$

$S = v_i t + \frac{1}{2}at^2$

$0 = u\sin\alpha\,(t_1+t_2) - \frac{1}{2}g(t_1+t_2)^2$

$\Rightarrow 0 = u\sin\alpha - \frac{1}{2}g(t_1+t_2)$

$t_1+t_2 = \frac{2u\sin\alpha}{g}$

using in (3), we get

$\frac{2u\sin\alpha}{g}\cdot u\cos\alpha = d(1+\frac{1}{e})$

$u^2\sin 2\alpha = gd(1+\frac{1}{e})$

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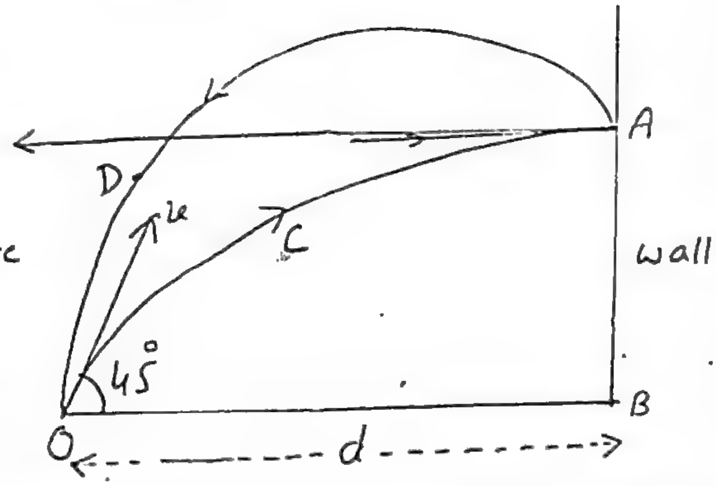
Problem#[16#] A ball is projected in a plane perpendicular to a smooth vertical wall so that after rebounding, it passes through the point of projection. If the point is at maximum distance (d) from the wall, then prove that V, the velocity of projection is given by

$V^2 = gd(1+\frac{1}{e})$, where e is the co-efficient of restitution.

Sol# Since (d) is the maximum distance of wall from the point of projection O, therefore the wall is at maximum range of ball and angle of projection will be 45°



Let t_1 be the time along path OCA and t_2 be time along path ADO.

$$d = t_1 \times V\cos 45^\circ \longrightarrow (1)$$

Horizontal velocity becomes $eV\cos 45^\circ$ after impact and hence

$$d = t_2 \cdot eV\cos 45^\circ$$

$$\frac{d}{e} = t_2 V\cos 45^\circ \longrightarrow (2)$$

Adding (1) & (2)

$$d + \frac{d}{e} = (t_1 + t_2) V\cos 45^\circ$$

$$d\left(1 + \frac{1}{e}\right) = (t_1 + t_2) \cdot \frac{1}{\sqrt{2}} V \longrightarrow (3)$$

Vertical Motion

Initial vertical velocity $= v_i = V\sin 45^\circ$

$$acc = -g$$

$$\text{time for motion} = t_1 + t_2$$

$$\text{total vertical distance} = 0$$

$$s = v_i t + \frac{1}{2} a t^2$$

$$0 = V\sin 45^\circ (t_1 + t_2) - \frac{1}{2} g (t_1 + t_2)^2$$

$$= \frac{1}{\sqrt{2}} V - \frac{1}{2} g (t_1 + t_2)$$

$$t_1 + t_2 = \frac{2}{\sqrt{2}} \frac{V}{g} = \frac{\sqrt{2} V}{g}$$

using in (3)

$$d\left(1+\frac{1}{e}\right) = \frac{\sqrt{2}V}{g} \cdot \frac{1}{\sqrt{2}} V$$

$$V^2 = gd\left(1+\frac{1}{e}\right)$$

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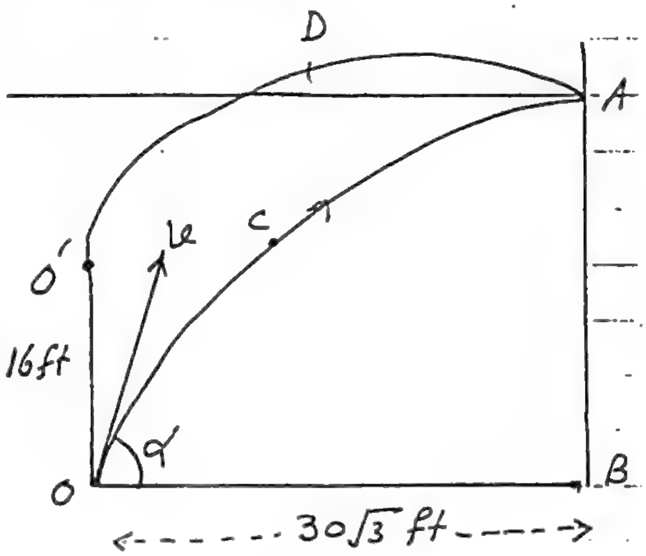
Problem # 16 # A ball is projected from a point O hits a vertical wall, rebounds and passes through the point 16 ft above O two seconds after projection. If the distance of the wall from O is $30\sqrt{3}$ ft. and the co-efficient of restitution is $3/5$, find the magnitude and direction of the initial velocity of the ball.

Find also the height above the level of O, of the point at which ball hits the ~~ball~~ wall

Sol # Let O be point of projection and V be the velocity of projection, α be angle projection. Let t_1 be time for motion along path OCA and t_2 be time for return journey along path ADO′.



Then $t_1 + t_2 = 2$ second given

Now

Horizontal distance = time × horizontal velocity

$$30\sqrt{3} = t_1 V\cos\alpha \longrightarrow (1)$$

on return

$$30\sqrt{3} = t_2 eV\cos\alpha$$

$$= \frac{3}{5} t_2 V\cos\alpha$$

$$50\sqrt{3} = t_2 V\cos\alpha \longrightarrow (2)$$

Adding (1) & (2)

$$80\sqrt{3} = (t_1+t_2)V\cos\alpha$$

$$80\sqrt{3} = 2V\cos\alpha$$

$$40\sqrt{3} = V\cos\alpha \longrightarrow (3)$$

Vertical Motion

Initial vertical velocity $= V\sin\alpha$

$acc = -g$

Vertical distace covered during total flight upto $O' = 16\,ft$

time $= t_1 + t_2 = 2\,sec = t$

$$S = V_i t + \frac{1}{2}at^2$$

$$16 = 2V\sin\alpha - \frac{1}{2}g \cdot 4$$

$$16 = 2V\sin\alpha - 2g$$

$$8 + g = V\sin\alpha \longrightarrow (4)$$

$$8 + 32 = V\sin\alpha$$

$$40 = V\sin\alpha \longrightarrow (4)$$

squaring and adding (3) & (4)

$$1600 + 4800 = V^2$$

$$6400 = V^2$$

$$V = \sqrt{6400} = 80\,ft/sec$$

Dividing (4) by (3)

$\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 60°$ with horizontal

putting value of V, α in (1)

$$30\sqrt{3} = t_1 \cdot 80 \cdot \frac{\sqrt{3}}{2}$$

$$t_1 = \frac{3}{4}\ \text{seconds}$$

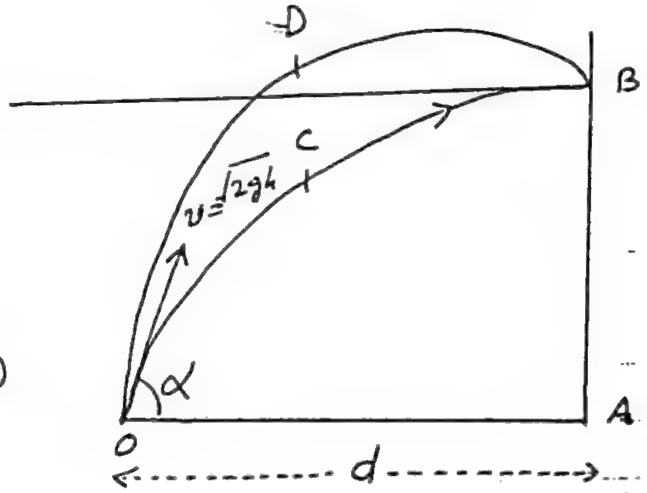
Height above O of point at which the ball hits the wall is vertical distance OO' covered in time $t_1 = \frac{3}{4}$ seconds. & is

$$\text{Distance} = V_0\sin\alpha\, t_1 - \frac{1}{2}g t_1^2$$

$$= 80\sin 30° \cdot \frac{3}{4} - \frac{1}{2}\cdot 32 \cdot \frac{9}{16} = 30 - 9$$

$$= 21\,ft$$

Problem # 17 # An imperfectly elastic ball is projected with velocity $v = \sqrt{2gh}$ at an angle α with the horizon, so that it strikes a vertical wall distant d from the point of projection and returns to the point of projection. Show that the co-efficient of restitution between the ball and wall is $\dfrac{d}{2h\sin 2\alpha - d}$.



Sol # Let O be the point of projection and t_1 be time for path OCB and t_2 be time for path BDO

$d = t_1 V\cos\alpha \longrightarrow ①$

on return

$d = t_2 eV\cos\alpha$

$\dfrac{d}{e} = t_2 V\cos\alpha \longrightarrow ②$

Adding ① & ②

$d\left(1+\frac{1}{e}\right) = (t_1+t_2)V\cos\alpha. \longrightarrow ③$

Vertical Motion #

Initial velocity $= v\sin\alpha$

acc $= -g$

time $= t_1+t_2$

total distance $=$ final vertical distance $= 0$

$s = v_i t + \frac{1}{2}at^2$

$0 = v\sin\alpha(t_1+t_2) - \frac{1}{2}g(t_1+t_2)^2$

$t_1+t_2 = \dfrac{2v\sin\alpha}{g}$

putting in ③

$d\left(1+\frac{1}{e}\right) = \dfrac{2v^2\sin\alpha\cos\alpha}{g} = \dfrac{(\sqrt{2gh})^2\sin 2\alpha}{g}$

$$d\left(1+\frac{1}{e}\right) = 2h\sin 2\alpha$$

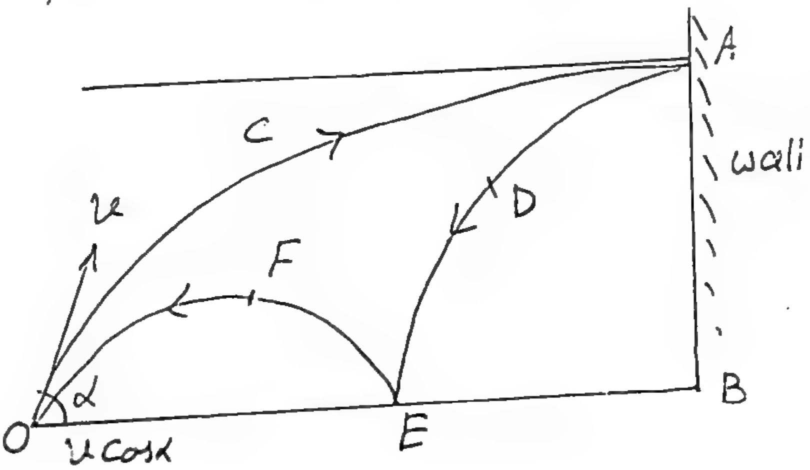
$$1+\frac{1}{e} = \frac{2h}{d}\sin 2\alpha$$

$$\frac{1}{e} = \frac{2h}{d}\sin 2\alpha - 1$$

$$= \frac{2h\sin 2\alpha - d}{d}$$

$$e = \frac{d}{2h\sin 2\alpha - d}$$

Problem # 18 # A particle is projected from a point in a smooth horizontal plane so as to strike a smooth vertical wall at right angles and after rebounding from the wall and once from the horizontal plane returns to the point of projection. Prove that the co-efficient of ~~(friction)~~ restitution is ½.



Sol: Let v be the velocity of projection at an angle α to the horizontal surface and t be time along path OCA.

Since the particle strikes the wall perpendicularly (i.e horizontal motion at point of contact), therefore it is at the peak point of its trajectory and time t from O to A is $t = \frac{v\sin\alpha}{g}$

OB = horizontal distance = time × horizontal velocity

$$= \frac{v\sin\alpha}{g} \cdot v\cos\alpha$$

$$OB = \frac{V^2 \sin\alpha \cos\alpha}{g} \longrightarrow (1)$$

Let the after impact with wall, the particle strikes at E in horizontal surface and after rebounding reaches at point of projection.
Let t_1, t_2 be times for paths ADE & EFO respectively

Vertical Motion

Initial vertical velocity $= V\sin\alpha$
acc $= -g$
time $= t_1 + t_2$
Vertical distance at $E = 0$

$$S = V_i t + \frac{1}{2}at^2$$
$$0 = V\sin\alpha(t_1+t_2) - \frac{1}{2}g(t_1+t_2)^2$$

$$t_1+t_2 = \frac{2V\sin\alpha}{g}$$

$$t_1 = \frac{2V\sin\alpha}{g} - t = \frac{2V\sin\alpha}{g} - \frac{V\sin\alpha}{g}$$

$$t_1 = \frac{V\sin\alpha}{g}$$

Vertical Velocity Gained upto E

$V_i = 0$ at A because particle was at peak of trajectory

$V_f = ?$
$t_1 = \frac{V\sin\alpha}{g}$
acc $= g$

$$V_f = V_i + a t_1$$
$$V_f = 0 + g\left(\frac{V\sin\alpha}{g}\right) = V\sin\alpha.$$

Motion after Rebounding from E

Vertical velocity after rebound from E $= eV\sin\alpha$

time $= t_2 = ?$ from E to O

Vertical distance covered $=$ zero

acc $= -g$

$$S = V_i t + \frac{1}{2}at^2$$

$$0 = eV\sin\alpha\, t_2 - \frac{1}{2}g t_2^2$$

$$\boxed{t_2 = \frac{eV\sin\alpha}{g}}$$

Thus

$$t_1 + t_2 = \frac{V\sin\alpha}{g} + \frac{2eV\sin\alpha}{g}$$

$$t_1 + t_2 = \frac{V\sin\alpha}{g}(2e+1)$$

OB $=$ horizontal distance $=$ horizontal velocity $\times$ time

at point A vertical velocity was zero and horizontal velocity was $V\cos\alpha$.

So after rebound from A horizontal velocity becomes $eV\cos\alpha$, which remains same upto back point O.

$$OB = (t_1 + t_2)\, eV\cos\alpha$$

$$= \frac{V\sin\alpha}{g}(2e+1)\, eV\cos\alpha.$$

$$= e\,\frac{V^2\sin\alpha\cos\alpha}{g}(1+2e) \longrightarrow (2)$$

By (1) & (2)

$$\frac{V^2\sin\alpha\cos\alpha}{g} = \frac{V^2\sin\alpha\cos\alpha}{g} \cdot e(1+2e)$$

$$1 = e(1+2e)$$

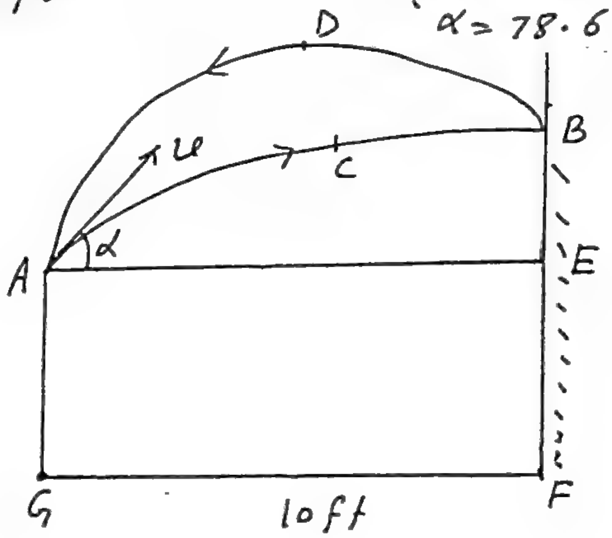
$$2e^2 + e - 1 = 0 \quad \Rightarrow (2e-1)(e+1) = 0$$

$e = \frac{1}{2}$ or $e = -1$ not acceptible.

$$\boxed{e = \frac{1}{2}}$$

Problem # 19 # A child throws a ball from a point A at some height from the horizontal with a speed of 50 ft/sec. It strikes a wall distant 10 ft and then returns exactly to point A. Determine the necessary angle α if the co-efficient of restitution in wall impact is .5 ($\alpha = 11.37°$ or $\alpha = 78.6°$)

Sol # Suppose u is the velocity of projection and α is the angle of projection. Let t_1 be time for path ACB and t_2 be time for path BDA



$10 = t_1 V\cos\alpha \longrightarrow (1)$

$\because$ horizontal velocity will be vertical to wall at B

$\therefore$ After rebound it becomes $eV\cos\alpha$

For return Journey

$10 = t_2 e V\cos\alpha$

$10 = .5\, t_2 V\cos\alpha \longrightarrow (3)$

$20 = t_2 V\cos\alpha \longrightarrow (2)$

Adding (1) & (2)

$30 = u\cos\alpha.(t_1+t_2) \longrightarrow (3)$

Now we consider vertical Motion.

Vertical Motion#

Initial velocity = $V\sin\alpha$

time = t_1+t_2

total vertical distance during whole fligh = 0

acc = $-g$

$$S = V_i t + \frac{1}{2}at^2$$

$$0 = V\sin\alpha\,(t_1+t_2) - \frac{1}{2}g\,(t_1+t_2)^2$$

$$t_1+t_2 = \frac{2V\sin\alpha}{g} \longrightarrow (4)$$

Dividing (1) by (2)

$$\frac{1}{2} = \frac{t_1}{t_2}$$

$$\Rightarrow t_2 = 2t_1 \longrightarrow (5)$$

putting in (4)

$$3t_1 = \frac{2V\sin\alpha}{g}$$

$$t_1 = \frac{2V\sin\alpha}{3g}$$

$$t_2 = \frac{4V\sin\alpha}{3g}$$

putting (4) in (3)

$$30 = \frac{2V^2\sin\alpha\cos\alpha}{g}$$

$$30g = V^2\sin 2\alpha$$

$$30g = (50)^2\sin 2\alpha$$

$$\sin 2\alpha = \frac{30\times 32}{(50)^2}$$

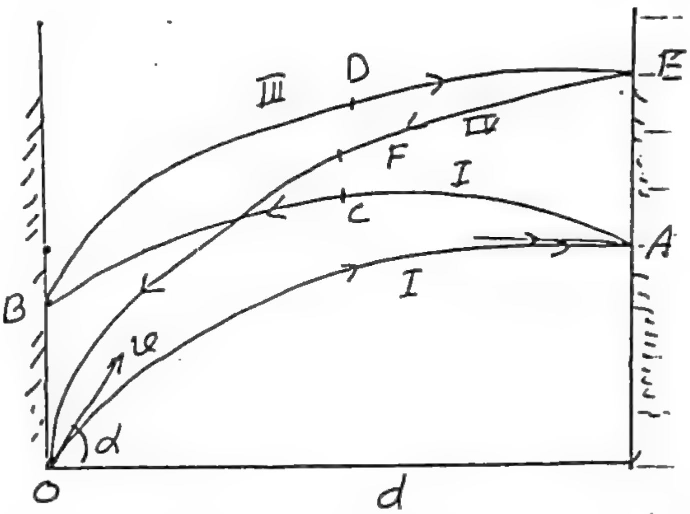
$$2\alpha = \sin^{-1}\left(\frac{30\times 32}{2500}\right)$$

$$\alpha = \frac{1}{2}\sin^{-1}\left(\frac{30\times 32}{2500}\right) =$$

Problem # 20 # An elastic ball is projected from a point at the foot of the one of the two smooth vertical walls so that after three impacts it may return to the point of projection. If the last impact be direct, show that

$$e^3+e^2+e=1$$

Sol # Let v be the velocity of projection and α be the angle of projection. Let d be distance between walls.
Let t_1 be time for path OA, t_2 for path ACB,



t_3 for path BDE and t_4 for path EFO.

For Path OA & path ACB

$$d = t_1 v\cos\alpha \longrightarrow (1)$$

After rebound horizontal velocity is $ev\cos\alpha$.
and

$$d = t_2 e v\cos\alpha$$

$$\frac{d}{e} = t_2 v\cos\alpha \longrightarrow (2)$$

For path BDE

Horizontal velocity $= e^2 v\cos\alpha$

$$d = t_3 e^2 v\cos\alpha$$

$$\frac{d}{e^2} = t_3 v\cos\alpha \longrightarrow (3)$$

For Path EFO #

Horizontal velocity $= e^3 v\cos\alpha$

$$d = t_4\, e^3\, u \cos\alpha$$

$$\frac{d}{e^3} = t_4\, u \cos\alpha \longrightarrow (4)$$

Adding (1), (2), (3) & (4)

$$d\left(1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\right) = (t_1+t_2+t_3+t_4)\, u \cos\alpha \longrightarrow (5)$$

Vertical Motion

Since ball returns to O, vertical distance is zero after time $t = t_1+t_2+t_3+t_4$

Initial vertical velocity $= V\sin\alpha$

time $= t_1+t_2+t_3+t_4$

Distance $= 0$

$$S = V_i t + \frac{1}{2} a t^2$$

$$0 = V\sin\alpha\,(t_1+t_2+t_3+t_4) + \frac{1}{2} g (t_1+t_2+t_3+t_4)^2$$

$$t_1+t_2+t_3+t_4 = \frac{2V\sin\alpha}{g}$$

putting This value in (5)

$$d\left(1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\right) = \frac{2V^2\sin\alpha\cos\alpha}{g} \longrightarrow (6)$$

Since Last impact is direct, so vertical Component of velocity is zero.

It means particle has attained maximum height $\frac{u^2\sin^2\alpha}{2g}$

Now we can find time t_4 for path EFO

Initial vertical velocity $= 0$

Distance $= \frac{u^2\sin^2\alpha}{2g}$

$t_4 = ?$

$$S = v_i t_4 + \frac{1}{2} g t_4^2$$

$$\frac{V^2 \sin^2\alpha}{2g} = 0 + \frac{1}{2} g \cdot t_4^2$$

$$t_4^2 = \frac{V^2 \sin^2\alpha}{g^2}$$

$$t_4 = \frac{V \sin\alpha}{g^2}$$

putting in (4)

$$\frac{d}{e^3} = \frac{V^2 \sin\alpha \cdot \cos\alpha}{g} \longrightarrow (7)$$

By (6) & (7)

$$d\left(1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\right) = \frac{2d}{e^3}$$

$$\Rightarrow \quad 1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3} = \frac{2}{e^3}$$

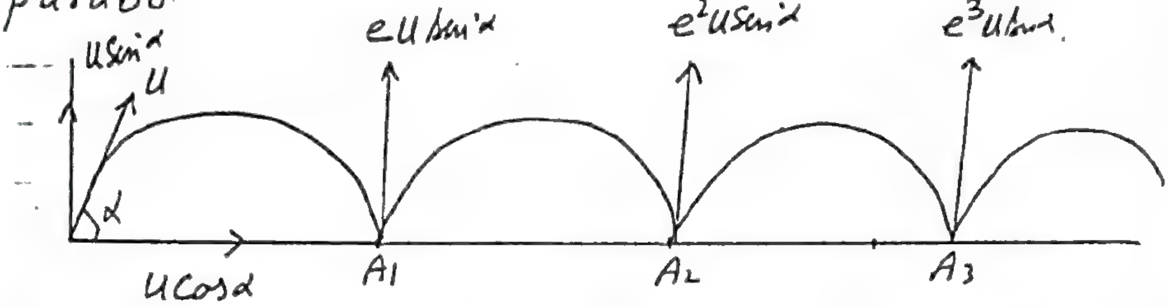
$$1+\frac{1}{e}+\frac{1}{e^2} = \frac{1}{e^3}$$

$$\Rightarrow e^3+e^2+e = 1 \quad \text{Ans}$$

Problem # 21 # From a point in a smooth horizontal plane a ball is projected with velocity u at an angle α to the horizon. Show that it will keep rebounding from the plane for a time $\frac{2u\sin\alpha}{g(1-e)}$ and will have range $\frac{u^2\sin 2\alpha}{g(1-e)}$, e being the Co-efficient of elasticity. Also prove that time taken is $\frac{1}{1-e}$ times of the time taken before 1st impact and range is $\frac{1}{1-e}$ Times of the range upto 1st impact.

Sol #. Suppose that ball is projected from O and it strikes the plane at points $A_1, A_2, A_3 \ldots$

The path of ball is parabolic. It strikes the plane, rebounds and so describes another parabola and again rebounds and describes a series of parabo[las]



The initial vertical component of velocity is $u\sin\alpha$ and vertical velocities of the ball at successive rebounds are $eu\sin\alpha$, $e^2u\sin\alpha$ etc

<u>Time of Flight before 1st impact</u> #

We consider vertical motion

Initial velocity $= u\sin\alpha$

acc $= -g$ time $= t_1$

Distance covered $= 0$

$$S = v_i t - \frac{1}{2}gt^2$$

$$0 = u\sin\alpha - \frac{1}{2}g t_1$$

$$t_1 = \frac{2u\sin\alpha}{g}$$

Time of flight After 1st rebound & Before 2nd Impact

$$0 = eu\sin\alpha\, t_2 - \frac{1}{2}g t_2^2$$

$$t_2 = \frac{e(2u\sin\alpha)}{g}$$

time of flight after 2nd impact and before 3rd impact is

$$t_3 = \frac{e^2(2u\sin\alpha)}{g}$$

So the Times for 1st, 2nd, 3rd, 4th etc parabolas are

$$\frac{2u\sin\alpha}{g},\ \frac{e(2u\sin\alpha)}{g},\ \frac{e^2(2u\sin\alpha)}{g},\ \frac{e^3(2u\sin\alpha)}{g}, \ldots$$

Total Time till rest

$$= \frac{2u\sin\alpha}{g} + \frac{e(2u\sin\alpha)}{g} + e^2\frac{(2u\sin\alpha)}{g} + \cdots$$

$$= \frac{2u\sin\alpha}{g}\left(1+e+e^2+e^3+\cdots\right)$$

$$= \frac{2u\sin\alpha}{g}\left(\frac{1}{1-e}\right)$$

$$= \left(\frac{1}{1-e}\right)\left(\frac{2u\sin\alpha}{g}\right)$$

$= \frac{1}{1-e}$ (time of flight before 1st impact.)

$1+e+e^2+e^3+\cdots$
It is infinite Geometric series with common ratio $r = e$.
$S_\infty = \frac{a_1}{1-r} = \frac{1}{1-e}$

Total Horizontal Range

$\because$ The plane is smooth
$\therefore$ There is no tangential (along plane) force and so Horizontal velocity $u\cos\alpha$ remain unchanged during whole of the motion.

Total Horizontal range = (Horizontal Velocity)(total time)

$$= u\cos\alpha \cdot \frac{1}{1-e}\,\frac{2u\sin\alpha}{g}$$

$$= \frac{1}{1-e}\left(\frac{u^2\sin 2\alpha}{g}\right)$$

$= \left(\frac{1}{1-e}\right)$ (Horizontal range before 1st impact after projection)

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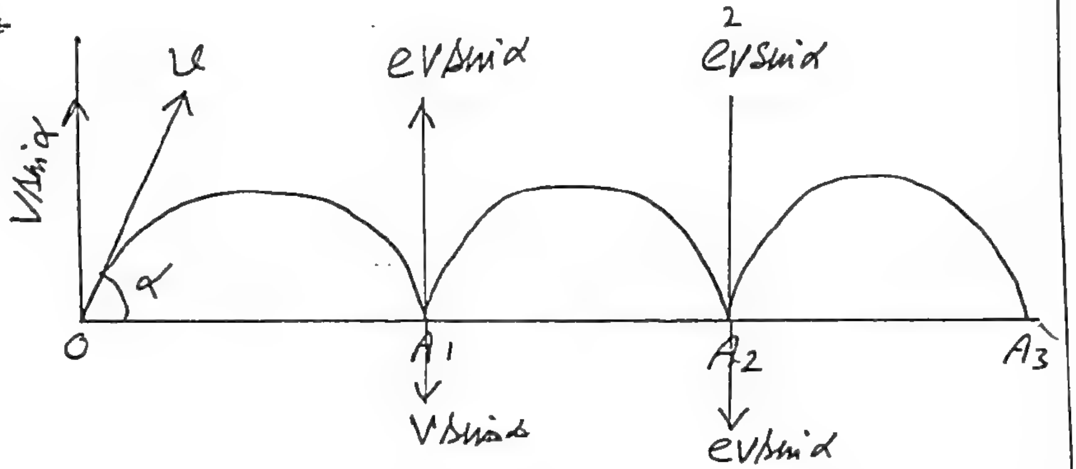
Problem # 22 # A ball is projected from a point in a smooth horizontal plane and makes two rebounds. Show that if the 3rd range is equal to the greatest height which balls attains before 1st rebound, then angle of projection is $\tan^{-1}(4e)$, e being the co-efficient of restitution #

Sol #



Let v be the velocity of projection and α be the angle of projection.

Greatest Height Before 1st Impact

We consider vertical motion.

The vertical velocity at the greatest height is zero. So greatest height is the vertical distance covered during time in which vertical velocity becomes zero

Initial vertical vel $= v_i = V\sin\alpha$

$acc = -g$

Distance $=$ greatest height $= H = ?$

Final vertical velocity $= V_f = 0$

$$2as = V_f^2 - V_i^2$$

$$-2gH = 0 - V^2\sin^2\alpha$$

$$H = \frac{V^2\sin^2\alpha}{2g} \longrightarrow (1)$$

For 3rd Horizontal Range

Since the horizontal plane is smooth, therefore there is no tangential force (force along the plane) and horizontal velocity $V\cos\alpha$ remains constant during whole the motion.

For time for 3rd range we consider vertical motion.

Initial vertical velocity $= V_i = e^2 V\sin\alpha$

$$acc = -g$$

Vertical distance covered $= 0$

$$s = V_i t + \frac{1}{2}at^2$$

$$0 = e^2 V\sin\alpha t - \frac{1}{2}gt^2$$

$$t = \frac{2e^2 V\sin\alpha}{g}$$

3rd horizontal range $= A_2A_3 =$ (Horizontal vel) (Time)

$$= (V\cos\alpha)\cdot\frac{2e^2V\sin\alpha}{g}$$

$$= \frac{2e^2V^2\sin\alpha\cos\alpha}{g} \longrightarrow (2)$$

But it is given that this range is equal to the greatest height attained before 1st impact. Therefore from (1) & (2)

$$\frac{V^2\sin^2\alpha}{2g} = \frac{2e^2V^2\sin\alpha\cos\alpha}{g}$$

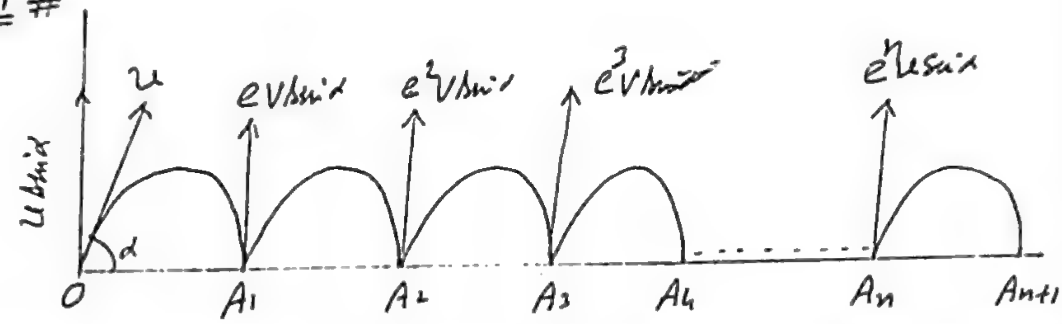
$$\frac{\sin\alpha}{2} = 2e^2\cos\alpha$$

$$\tan\alpha = 4e^2$$

$$\alpha = \tan^{-1}(4e^2)$$

Problem#23)# A ball is projected from a point in a smooth horizontal plane and makes ~~two~~ n rebounds. Show that if the $(n+1)$th range is equal to the greatest height which ball attains before 1st rebound, then angle of projection is $\tan^{-1}(4e^n)$, e being the co-efficient of restitution.

Sol #



Let v be the velocity of the projection and α be the angle of projection.

The greatest height reached before 1st rebound is

$$H = \frac{V^2 \sin^2\alpha}{2g} \longrightarrow (1)$$

For $(n+1)$th range A_nA_{n+1} #

Since the horizontal plane is smooth, therefore the horizontal velocity remains constant.

For time of this range we consider vertical motion.

Initial vertical velocity $= e^n V \sin\alpha$

$acc = -g$

vertical distance covered $= 0$

time $= t = ?$

$$S = V_i t + \frac{1}{2} a t^2$$

$$0 = e^n V \sin\alpha . t - \frac{1}{2} g t^2$$

$$t = \frac{2 e^n V \sin\alpha}{g}$$

$(n+1)$ the range $= A_n A_{n+1} = $ (Horizontal vel) (time).

$$= (V\cos\alpha)\cdot\frac{2e^n V\sin\alpha}{g}$$

$$= \frac{2e^n V^2 \sin\alpha\cos\alpha}{g} \longrightarrow ②$$

But it is given that the $(n+1)$th horizontal range is equal to the greatest height before 1st rebound. Therefore by ① & ②

$$\frac{V^2\sin^2\alpha}{2g} = \frac{2e^n V^2\sin\alpha\cos\alpha}{g}$$

$$\tan\alpha = 4e^n$$

$$\alpha = \tan^{-1}(4e^n)$$

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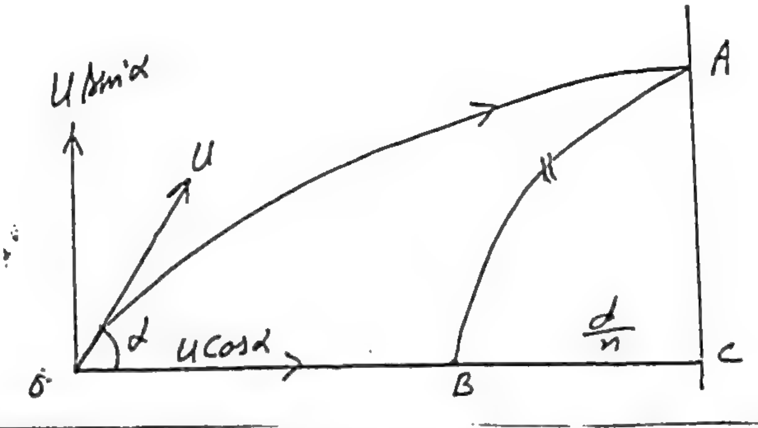
Problem # 24 # A particle is projected from a smooth horizontal floor/ground so as to strike a smooth vertical wall at distance d. After impact the particle strikes the floor/ground at a point distant $\frac{1}{n}$ $(n>1)$ of the way between the point of projection and wall. Show that the angle of projection is

$$\frac{1}{2}\sin^{-1}\left[\frac{(1+ne)dg}{2nu^2}\right],$$ where e is the co-efficient of restitution between particle and wall.

Sol #



Let α be the angle of projection and O be the point of projection. Suppose after impact with the wall the particle strikes the ground at B such that

$$BC = \frac{d}{n}$$

Let t_1 be the time for path OA and t_2 be the time for path AB.

Horizontal distance = (Horizontal vel) (Time)

$$d = u\cos\alpha\, t_1 \longrightarrow ①$$

Since wall is smooth, therefore on impact $u\sin\alpha$ remain same but $u\cos\alpha$ becomes $eu\cos\alpha$ and we have

$$\frac{d}{n} = eu\cos\alpha\, t_2$$

$$\frac{d}{ne} = u\cos\alpha\, t_2 \longrightarrow ②$$

Adding ① & ②

$$d + \frac{d}{ne} = (t_1+t_2)\, u\cos\alpha$$

$$d\left(\frac{ne+1}{ne}\right) = (t_1+t_2)\, u\cos\alpha \longrightarrow ③$$

Vertical Motion #

Initial velocity $= u\sin\alpha$

acc $= -g$

time $= t_1+t_2$

Vertical distance covered upto B = 0

$$S = v_i t + \frac{1}{2}at^2$$

$$0 = u\sin\alpha\,(t_1+t_2) - \frac{1}{2}g\,(t_1+t_2)^2$$

$$\Rightarrow t_1+t_2 = \frac{2u\sin\alpha}{g}$$

putting in ③

$$\frac{d(ne+1)}{ne} = \frac{u^2\sin 2\alpha}{g}$$

$$\Rightarrow \quad \frac{gd(1+ne)}{ne\,u^2} = \sin 2\alpha$$

$$\alpha = \frac{1}{2}\sin^{-1}\left[\frac{gd(1+ne)}{ne\,u^2}\right]$$

If the particle strikes the ground mid way then $n = 2$ and angle of projection will be

$$\alpha = \frac{1}{2}\sin^{-1}\left[\frac{gd(1+3e)}{2e\,u^2}\right]$$

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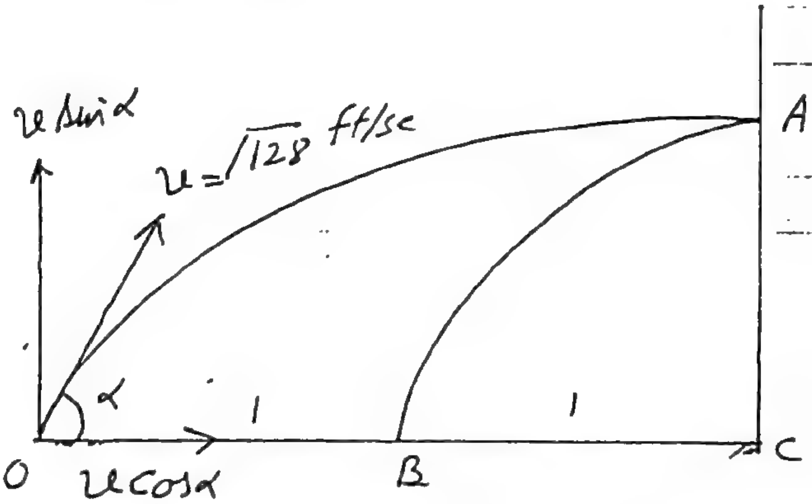
Problem # 25 # A particle is projected from some point with a velocity of $\sqrt{128}$ ft/sec to strike a smooth vertical wall at distance 4ft. After the impact with wall the particle strikes the ground mid-way between the point of projection and wall. If $e = \frac{1}{2}$ is the co-efficient of restitution between particle and wall, then find the angle of projection.

Sol #



Let O be point of projection and α be angle of projection. Suppose after impact with the wall the particle strikes the ground at point B such that BC = 2 ft

Let t_1 be time for path OA and t_2 be time for path AB.

$$2 = V\cos\alpha\; t_1 \longrightarrow (1)$$

After impact with wall horizontal velocity becomes $eV\cos\alpha$

$$1 = BC = eV\cos\alpha \cdot t_2$$

$$1 = \frac{1}{2} V\cos\alpha \cdot t_2$$

$$2 = V\cos\alpha \cdot t_2 \longrightarrow (2)$$

Adding (1) & (2)

$$4 = V\cos\alpha (t_1 + t_2) \longrightarrow (3)$$

Vertical Motion#

Initial vertical velocity $= V\sin\alpha$

Time upto $B = t_1 + t_2$

$acc = -g$

Vertical distance covered upto $B = 0$

$$S = V_i t + \frac{1}{2} a t^2$$

$$0 = V\sin\alpha (t_1 + t_2) - \frac{1}{2} g (t_1 + t_2)^2$$

$$0 = V\sin\alpha - \frac{1}{2} g (t_1 + t_2)$$

$$t_1 + t_2 = \frac{2V\sin\alpha}{g}$$

putting in (3)

$$4 = \frac{V^2 \sin 2\alpha}{g}$$

$$4 = \frac{(128)\sin 2\alpha}{32}$$

$$128 = 128 \sin 2\alpha$$

$$\Rightarrow \sin 2\alpha = 1$$

$$2\alpha = 90^\circ$$

$$\alpha = 45^\circ \text{ Ans}$$

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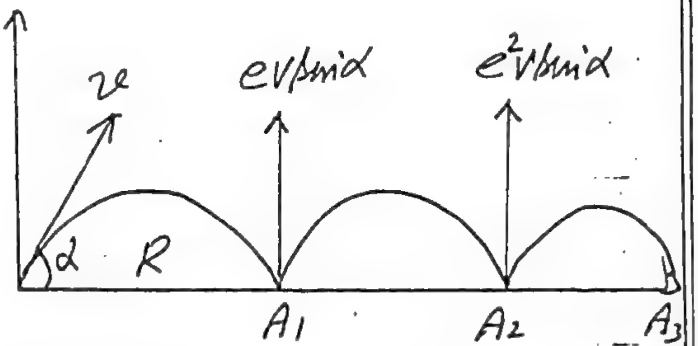
Problem # 26 # A particle of elasticity e is projected in a direction inclined to vertical and bounces on smooth horizontal plane. The range of one rebound is R, find ranges of next two rebounds and total ranges of three rebounds #

Sol # Let v be the velocity of projection and α be angle of projection with horizontal



Time of flight befor 1st rebound $= t_1 = \dfrac{2v\sin\alpha}{g}$

$$\text{Range} = \frac{2v^2\sin\alpha\cos\alpha}{g}$$

Time of fligt after 1st rebound and before 2nd rebound $= \dfrac{2(e\sin\alpha)}{g}$

$$\text{Second range} = v\cos\alpha \times \frac{2ev\sin\alpha}{g}$$

$$= e\left(\frac{2v^2\sin\alpha\cos\alpha}{g}\right)$$

$$= eR$$

Similarly Third range $= e^2R$

Total range $= R + eR + e^2R$

$= R(1 + e + e^2)$

Problem # 27 # (Challege Exercise for Professor readers)

A particle of elasticity e goes bouncing down on an endless flight of stairs each of height h and breadth b. It hits each stair at an exactly

similar point at same angle and with same speed. Find the angle of impact and show that the normal velocity of impact is $\sqrt{\dfrac{2gh}{1-e^2}}$

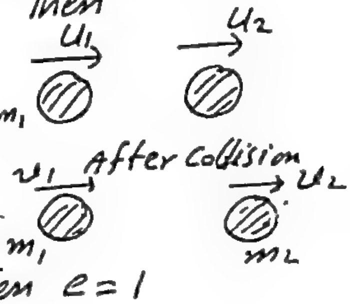
Ans $\tan^{-1}\left[\dfrac{b(1-e)}{2h}\right]$ with vertical

Problem# 28)# (Challege Exercise for Prof. reader)
Determine the co-efficient of restitution e which will allow the ball to bounce down the endless flight of stairs. The tread and rises dimensions, d & h respectively are same for each step and the ball bounced the same height h' above each step. What horizontal u_x is required so that the ball lands in the centre of each tread. Ans $e = \sqrt{\dfrac{h'}{h'+h}}$

$$V_x = \frac{\sqrt{\frac{g}{2}}\, d}{\sqrt{h'} + \sqrt{h'+h}}$$

Direct Imact of Moveable Objects

If elastic particles or spheres of masses m_1, m_2 moving to after other along the same straight line through their centres with velocity u_1 & u_2 collide and v_1, v_2 are their velocities after collision, Then



$$v_1 = \left(\frac{m_1 - e m_2}{m_1+m_2}\right)u_1 + \frac{(1+e)m_2 u_2}{m_1+m_2}$$

$$v_2 = \left(\frac{m_2 - e m_1}{m_1+m_2}\right)u_2 + \frac{(1+e)m_1 u_1}{m_1+m_2}$$

If spheres are perfectly elastic, then $e = 1$ and

$$v_1 = \left(\frac{m_1 - m_2}{m_1+m_2}\right)u_1 + \frac{2m_2 u_2}{m_1+m_2}$$

$$v_2 = \left(\frac{m_2 - m_1}{m_1+m_2}\right)u_2 + \frac{2m_1 u_1}{m_1+m_2}$$

2) # Impulse of the blow on each sphere

= Change in momentum of either

$$= \frac{m_1 m_2}{m_1+m_2}(1+e)(u_2-u_1) \quad \text{(magnitude)}$$

3) # Distance covered relative to one another after impact in time t

= time × relative velocity of seperation

$$= t(v_2-v_1)$$

$$= te(u_2-u_1) \qquad \because \frac{v_1-v_2}{u_1-u_2} = -e$$

$$\Rightarrow v_2-v_1 = e(u_2-u_1)$$

4) # Loss in K·E during impact

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(1-e^2)(u_2-u_1)^2$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(1-e^2)(u_1-u_2)^2$$

where velocities are taken as +ve in arbitrarily one direction and −ve in the opposite direction.

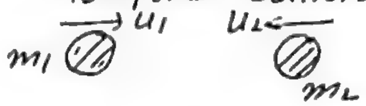
Impulse of Blow in Direct Collision

Problem # 29 # ☆ (2002) Two smooth spheres of masses m_1 & m_2 moving with velocities u_1 & u_2 respectively collide directly. If e is the co-efficient of elasticity, show that the momentum lost by one and gained by by other is $\frac{m_1 m_2}{m_1+m_2}(1+e)(u_1+u_2)$ OR momentum lost by one and gained by other is $\frac{m_1 m_2}{m_1+m_2}(1+e)$ time their relative velocity before impact or impulse of blow on each is $\frac{m_1 m_2}{m_1+m_2}(1+e)(u_1+u_2)$.

Sol # Let u_1 & u_2 be velocities of spheres of masses m_1 & m_2 before collision and v_1, v_2 be velocities after collision. Suppose the spheres are moving in opposite directions before (~~directions~~) collision as shown

Then taking velocity towards right +ve, we have

Before collision



$$v_1 = \left(\frac{m_1 - e m_2}{m_1+m_2}\right)u_1 + \frac{(1+e)m_2}{m_1+m_2}(-u_2)$$

Momentum of m_1 before collision $= m_1 u_1$

Momentum of m_2 after collision $= m_1 v_1$

Loss in momentum of m_1 or momentum transferred by m_1 or impulse received by m_1

$$= m_1 u_1 - m_1 v_1$$

$$= m_1 u_1 - m_1\left[\left(\frac{m_1 - e m_2}{m_1+m_2}\right)u_1 - \frac{(1+e)m_2 u_2}{m_1+m_2}\right]$$

$$= m_1 u_1 - m_1\left[\frac{(m_1 - e m_2)u_1 - (1+e)m_2 u_2}{m_1+m_2}\right]$$

$$= \frac{m_1 u_1(m_1+m_2) - m_1(m_1 - e m_2)u_1 + (1+e)m_1 m_2 u_2}{m_1+m_2}$$

$$= \frac{\cancel{m_1^2 u_1} + m_1 m_2 u_1 - \cancel{m_1^2 u_1} + e m_1 m_2 u_1 + (1+e)m_1 m_2 u_2}{m_1+m_2}$$

$$= \frac{(1+e)m_1 m_2 u_1 + (1+e)m_1 m_2 u_2}{m_1+m_2}$$

$$= \frac{(1+e)m_1 m_2}{m_1+m_2}(u_1+u_2)$$

$$= \frac{(1+e)m_1 m_2}{m_1+m_2}[u_1 - (-u_2)]$$

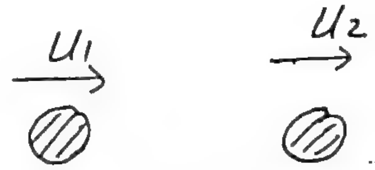
$$= \frac{(1+e)\,m_1 m_2}{m_1+m_2}\left(\text{relative velocity of spheres before collision}\right)$$

= Momentum received by sphere of mass m_2

Problem # 30 # Two spheres of masses m_1 & m_2 collide directly with velocities u_1 & u_2 respectively. Show that the momentum lost by one and gained by other is

$$\frac{m_1 m_2}{m_1+m_2}(1+e).(u_1-u_2)$$

Sol # Suppose the spheres are moving in opposite directions as shown. Taking velocity toward right +ve, we have



Before collision

$$v_1 = \left(\frac{m_1 - e m_2}{m_1+m_2}\right)u_1 + \frac{(1+e)\,m_2}{m_1+m_2}u_2$$

Momentum of m_1 before collision = $m_1 u_1$
Momentum of m_1 after collision = $m_1 v_1$
Momentum lost by m_1 = $m_1 u_1 - m_1 v_1$

$$= m_1 u_1 - m_1\left[\frac{(m_1 - e m_2)u_1 + (1+e)\,m_2 u_2}{m_1+m_2}\right]$$

$$= \frac{m_1 u_1(m_1+m_2) - m_1(m_1 - e m_2)u_1 - m_1(1+e)\,m_2 u_2}{m_1+m_2}$$

$$= \frac{\cancel{m_1^2 u_1} + m_1 m_2 u_1 - \cancel{m_1^2 u_1} + e\,m_1 m_2 u_1 - (1+e)\,m_1 m_2 u_2}{m_1+m_2}$$

$$= \frac{(1+e)\,m_1 m_2 (u_1-u_2)}{m_1+m_2} \quad \text{(Proved)}$$

Problem # 31 # Two smooth spheres of masses m_1 & m_2 moving with velocities 4 m/sec and 1 m/sec along the same straight line collide. If the sphers are moving in opposite directions and loss of K.E during collision is numerically equal to the momentum gained by one and lost by the other, then find coefficient of elasticity.

Sol # Let $u_1 = 4$ m/sec $u_2 = 1$ m/sec

Since spheres are moving in opposite direction, therefore

$$\text{Loss in K.E} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1-e^2)(u_1+u_2)^2$$

Momentum gained by one and lost by other

$$= \frac{(1+e)\,m_1 m_2}{m_1+m_2}(u_1+u_2)$$

According to given condition

$$\frac{(1+e)\,m_1 m_2}{m_1+m_2}(u_1+u_2) = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1-e^2)(u_1+u_2)^2$$

$$1 = \frac{1-e}{2}(u_1+u_2)$$

$$u_1+u_2 = \frac{2}{1-e}$$

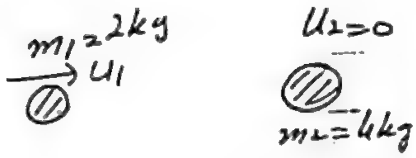
$$4+1 = \frac{2}{1-e}$$

$$5(1-e) = 2$$

$$-5e = -3$$

$$e = 3/5$$

Problem # 32 # A smooth sphere of mass 2 kg is moving with speed 3 m/sec on a horizontal plane when it collides with a stationary smooth sphere of equal size but of mass 4 kg. If the co-efficient of restitution between spheres is ½



Sol # Let $m_1 = 2kg$
$m_2 = 4kg$

Suppose u_1 is velocity of m_1 & u_2 is velocity of m_2 before collision.

Then $u_1 = 3 m/sec$ $\quad u_2 = 0 m/sec$

Let v_1 & v_2 be their velocities after collision.

$$v_1 = \left(\frac{m_1 - e m_2}{m_1 + m_2}\right) u_1 + \frac{(1+e) m_2 u_2}{m_1 + m_2}$$

$$= \frac{2 - \frac{1}{2}(4)}{2+4}(3) + 0 \qquad \because u_2 = 0$$

$$= 0$$

$\Rightarrow$ Colliding spheres comes to rest after collision.

$$v_2 = \left(\frac{m_2 - e m_1}{m_1 + m_2}\right) u_2 + \frac{(1+e) m_1 u_1}{m_1 + m_2}$$

$$= 0 + \frac{(1+\frac{1}{2}) \cdot 2 \cdot 3}{6}$$

$$= 3/2 = 1.2 \; m/sec.$$

$\Rightarrow$ sphere of mass m_2 (target sphere) starts moving with velocity of $1.2 m/sec$

Problem* # 33 # (2006). Three perfectly elastic balls of masses $m, 2m, 3m$ are placed in a straight line. The 1st impinge directly on 2nd with velocity u and then the 2nd impinges on third. Find the velocity of third ball after impact;

If the masses of 2nd and 3rd balls are m_1 & m_2, then show that 3rd ball will move after impact with velocity u if

$$(m + m_1)(m_1 + m_2) = 4 m m_1$$

Sol **Impact Between m & $2m$.**

Let u & $v = 0$ be velocities before collision and u_1', v_1' be there velocities after collision.

Then

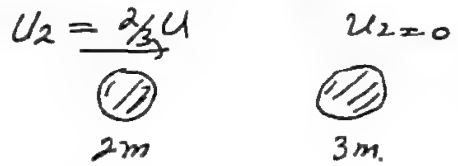
$$u_1' = \left(\frac{m-2m}{m+2m}\right)u + 0 = -\tfrac{1}{3}u.$$

$$v_1' = \frac{2m-m}{3m}v_1 + \frac{(1+e)\,m}{3m}u.$$

$$= 0 + \frac{(1+1)}{3}u = \tfrac{2}{3}u.$$

Thus after collision m comes back with velocity $\frac{1}{3}u$ and $2m$ strarts to move with velocity $\frac{2}{3}u$.

Impact Between $2m$ & $3m$ #



Before impact let velocities be $u_2 = \frac{2}{3}u$ & $v = 0$.

Let u_2', v_2' be velocities after impact. Then

$$v_2' = \left(\frac{3m-2m}{2m+3m}\right)u_2 + \frac{(1+e)\,2m}{2m+3m}u_2.$$

$$= 0 + \frac{(1+1)\cdot 2}{5}\cdot\tfrac{2}{3}u$$

$$= \tfrac{8}{15}u$$

Thus velocity of 3rd ball is $\frac{8}{15}u$.

When masses of 2nd and 3rd balls are m_1, m_2. Then we have

Impact Between m & m_1

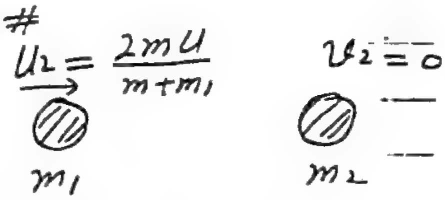
Let $u_1 = u$ & $v_1 = 0$
be velocities before impact
and u_1', v_1' be velocities after impact.
Then

$$v_1' = \left(\frac{m_1 - m}{m_1 + m}\right) v_1 + \frac{(1+e)mu}{m+m_1}$$

$$= 0 + \frac{(1+1)mu}{m+m_1}$$

$$= \frac{2mu}{m+m_1}$$

Impact Between m_1 & m_2



Let $u_2 = \dfrac{2mu}{m+m_1}$ & $v_2 = 0$.
be velocities before impact
and u_2', v_2' be velocities after impact.
Then

$$v_2' = \frac{m_2 - m_1}{m_1 + m_2} v_2 + \frac{(1+e)m_1 u_2}{m_1 + m_2}$$

$$= 0 + \frac{(1+1)m_1}{m_1+m_2} \times \frac{2mu}{m+m_1}$$

$$= \frac{4mm_1 u}{(m_1+m_2)(m+m_1)}$$

Now 3rd ball of mass m_2 will move with vel u if

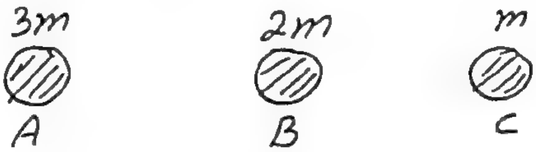
$$v_2' = u$$

if $$\frac{4m_1 m u}{(m_1+m_2)(m+m_1)} = u$$

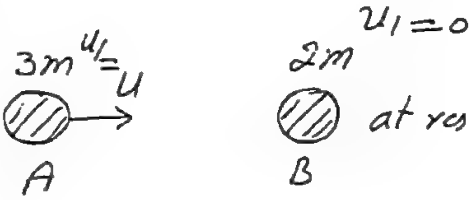
if $$4m_1 m = (m_1+m_2)(m+m_1)$$ proved.

Problem # 34 # Three perfectly elastic spheres A, B, C have masses 3m, 2m, m respectively. They are lying in a straight line on a horizontal plane and A is projected with speed u to collide directly with B which goes to collide directly with C. Find the speed of each sphere after 2nd impact. Explain why there will be no further impact
(Ans $\frac{u}{5}$, $\frac{2u}{5}$, $\frac{8u}{5}$)

Sol



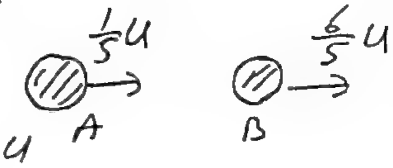
Impact Between A & B #



Let $u_1 = u$ velocity of A and $v_1 = 0$ be velocity of B before collision.
Let u_1', v_1' be velocities after collision.

$$u_1' = \left(\frac{3m - e\cdot 2m}{3m+2m}\right)u + \frac{(1+e)2m}{3m+2m}v_2$$

$$u_1' = \frac{3m-2m}{5m}u + 0 = \frac{u}{5}$$

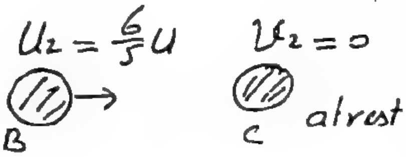


$$v_1' = \frac{2m - 3me}{5m}v_1 + \frac{(1+e)3mu}{5m}$$

$$= 0 + \frac{6}{5}u$$

$$v_1' = \frac{6}{5}u$$

Impact Between B & C #



Let $u_2 = \frac{6}{5}u$ & $v_2 = 0$

be velocities of B & C before collision.
and u_2', v_2' be their velocities after collision.

$$u_2' = \left(\frac{2m-m}{2m+m}\right)u_2 + \frac{(1+e)m}{2m+m}v_2$$

$$= \frac{m}{3m}\cdot\frac{6}{5}u + 0$$

$$= \frac{2}{5}u$$

$$v_2' = \left(\frac{m-2m}{3m}\right)v_2 + \frac{(1+e)\cdot 2m}{3m}u_2.$$

$$= 0 + \frac{2\cdot 2m}{3m}\cdot\frac{6}{5}u$$

$$v_2' = \frac{8}{5}u$$

Thus velocities of A, B and C after 2nd impact are

$$\frac{1}{5}u,\ \frac{2}{5}u,\ \frac{8}{5}u$$

Since $\frac{1}{5}u < \frac{2}{5}u < \frac{8}{5}u$

Therefore no will take over the other and There will be no further impact

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Problem # 35 # If the masses of two balls be as 3 : 1 and their respective velocities before impact be as 1 : 3 in opposite direction. Show That if the co-efficient of restitution is 7/8, then each ball moves back after impact with 7/8 of the original velocity.

Sol # Let m_1 & m_2 be the masses of balls.

Then $m_1 : m_2 = 3 : 1$

$$\Rightarrow \frac{m_1}{m_2} = \frac{3}{1} \Rightarrow m_1 = 3m_2$$

Let u_1 & u_2 be the velocities of balls before collision and v_1 & v_2 be their velocities after collision.

$u_1 : u_2 = 3:1$

$\Rightarrow \frac{u_1}{u_2} = \frac{3}{1}$

$\Rightarrow u_1 = 3u_2$

taking velocity towards right +ve.

$$v_1 = \left(\frac{m_1 - em_2}{m_1+m_2}\right)u_1 + \frac{(1+e)m_2(-u_2)}{m_1+m_2}$$

$$= \frac{3m_2 - \frac{7}{8}m_2}{(3+1)\,m_2}u_1 - \frac{(1+\frac{7}{8})\,m_2}{(3+1)\,m_2}u_2$$

$$= \frac{17/8}{4}u_1 - \frac{45}{32}u_1$$

$$= \frac{17}{32}u_1 - \frac{45}{32}u_1 = -\frac{28}{32} = -\frac{7}{8}u_1$$

$\Rightarrow m_1$ moves back with velocity $\frac{7}{8}u_1$

$$v_2 = \left(\frac{m_2 - em_1}{m_1+m_2}\right)u_2 + \frac{(1+e)m_1u_1}{m_1+m_2}$$

$$= \frac{7}{8}u_2 = -\frac{7}{8}(-u_2)$$

$\Rightarrow m_2$ moves back with speed $\frac{7}{8}u_2$

Problem # 36 # If the masses of two balls be as $r:1$ $(r>1)$ and their respective velocities before impact be as $1:r$ in opposite directions and $e = \frac{2r+1}{2(r+1)}$, then show that each ball will after direct impact move back with $\frac{2r+1}{2(r+1)}$ th of the original velocity

Sol# Let m_1, m_2 be masses of two balls A & B and u_1, u_2 be their velocities before impact. Then

$$m_1 : m_2 = r : 1$$

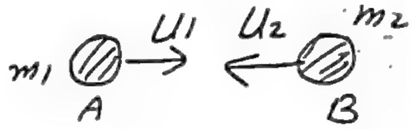
$$\frac{m_1}{m_2} = \frac{r}{1}$$

$$m_1 = r m_2$$

Also $u_1 : u_2 = 1 : r$

$$\Rightarrow \frac{u_1}{u_2} = \frac{1}{r} \Rightarrow u_2 = r u_1$$

$$e = \frac{2r+1}{2(r+1)}$$



Taking velocities towards right as +ve.
Let v_1 & v_2 be velocities after Collision.

$$v_1 = \frac{m_1 - e m_2}{m_1+m_2} u_1 + \frac{(1+e)\, m_2}{m_1+m_2}(-u_2)$$

$$= \frac{r m_2 - \frac{2r+1}{2(r+1)} \cdot m_2}{(r+1) m_2} u_1 - \frac{\left[1 + \frac{2(r)+1}{2(r+1)}\right] m_2\, r u_1}{(r+1) m_2}$$

$$= \frac{2r^2 + \cancel{2r} - \cancel{2r} - 1}{2(r+1)^2} u_1 - \frac{[2r+2+2r+1] r u_1}{2(r+1)^2}$$

$$= \frac{2r^2-1}{2(r+1)^2} u_1 - \frac{(4r^2+3r)}{2(r+1)^2} u_1$$

$$= -\frac{(2r^2+3r+1)}{2(r+1)^2} u_1$$

$$= -\frac{[2r^2+2r+r+1]}{2(r+1)^2} u_1$$

$$= -\frac{[2r(r+1)+1(r+1)]}{2(r+1)^2} u_1$$

$$= -\frac{(2r+1)(r+1)}{2(r+1)^2} u_1 = -\frac{2r+1}{2(r+1)} u_1$$

Similarly

$$v_2 = +\frac{2r+1}{2(r+1)}u_2$$

Thus each ball moves back with $\frac{2r+1}{2(r+1)}$ th. of the original velocity

Problem # 37 # A ball A, moving with velocity u impinges directly on an equal ball B moving with velocity v in opposite direction. If A is brought to rest by impact, then show that

$$u : v = 1+e : 1-e$$

Sol # Let $u_1 = 0$, v_1 be velocities after impact. Then

$$v_1 = \frac{m-em}{2m}(-v) + \frac{(1+e)m}{2m}u$$

$$= \left(\frac{1-e}{2}\right)(-v) + \left(\frac{1+e}{2}\right)u \rightarrow ①$$

$$0 = \frac{m-em}{2m}u + \left(\frac{1+e}{2}\right)(-v)$$

$$= \left(\frac{1-e}{2}\right)u - \left(\frac{1+e}{2}\right)v$$

$$\Rightarrow 0 = (1-e)u - (1+e)v$$

$$\Rightarrow (1+e)v = (1-e)u$$

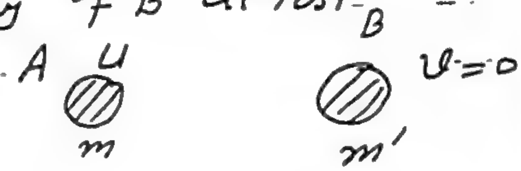
$$\Rightarrow \frac{u}{v} = \frac{1+e}{1-e}$$

Problem # 38 # A sphere impinges directly on an equal sphere at rest if the co-efficient of restitution is e. Show that their velocities after impact are as $1-e/1+e$

If the mass of 1st be m and that of 2nd be

2nd is m', show that the 1st can not have its velocity reversed if $m > em'$

Sol# 3 Let u be velocity of sphere A and $v = 0$ be the velocity of B at rest. let u_1, v_1 be the velocities after impact



$$u_1 = \frac{m - em}{2m} u + 0$$

$$= \left(\frac{1-e}{2}\right) u \longrightarrow ①$$

$$v_1 = \frac{m - em}{2m}(0) + \frac{(1+e)m}{2m} u$$

$$= \left(\frac{1+e}{2}\right) u \longrightarrow ②$$

$$\frac{u_1}{v_1} = \frac{1-e}{1+e}$$

$$u_1 = \frac{m - em'}{m + m'} u \longrightarrow ③$$

Now if $m > em'$, then u_1 is +ve and 1st can not have its velocity reversed.

Problems Related to energy lost.

Problem# 39# (2001) Two smooth equal spheres moving with velocities u_1 & u_2 in the direction of line joining their centres, impinge directly. If the co-efficient of restitution is $\frac{1}{2}$. Show exactly half the energy is lost in collision if u_1 & u_2 are in ratio $1+\sqrt{2} : 1-\sqrt{2}$

Sol# It is given that

$$u_1 : u_2 = 1+\sqrt{2} : 1-\sqrt{2}$$

$$\Rightarrow u_1 = \frac{1+\sqrt{2}}{1-\sqrt{2}} u_2$$

Total energy lost before collision

$$= \frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2$$

$$= \frac{1}{2}m\left[u_1^2 + u_2^2\right]$$

$$= \frac{1}{2}m\left[\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)^2 u_2^2 + u_2^2\right]$$

$$= \frac{1}{2}m\left[\frac{(1+\sqrt{2})^2 + (1-\sqrt{2})^2}{(1-\sqrt{2})^2}\right]u_2^2$$

$$= \frac{1}{2}m \cdot \frac{6}{(1-\sqrt{2})^2}\, u_2^2$$

$$= \frac{3m}{(1-\sqrt{2})^2}\, u_2^2 \longrightarrow (1)$$

K.E lost in collision $= \frac{1}{2}\left(\frac{mm}{m+m}\right)(1-e^2)(u_1-u_2)^2$

$$= \frac{1}{2}\cdot\frac{1}{2m}m^2\left(1-\frac{1}{4}\right)\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}u_2 - u_2\right)^2$$

$$= \frac{1}{4m}m^2\left(\frac{3}{4}\right)\left(\frac{1+\sqrt{2}-(1-\sqrt{2})}{1-\sqrt{2}}\right)^2 u_2^2$$

$$= \frac{1}{4m}\cdot\frac{3m^2}{4}\,\frac{(2\sqrt{2})^2}{(1-\sqrt{2})^2}\, u_2^2$$

$$= \frac{3m^2}{16m}\cdot\frac{8}{(1-\sqrt{2})^2}\, u_2^2$$

$$= \frac{1}{2}\left[\frac{3m}{(1-\sqrt{2})^2}u_2\right]^2 = \frac{1}{2}(\text{total k.E befor collision})$$

Note # Here $\frac{u_1}{u_2} = \frac{1+\sqrt{2}}{1-\sqrt{2}} = -\frac{1+\sqrt{2}}{\sqrt{2}-1}$

$$\Rightarrow u_1 = -\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right) u_2$$

$\Rightarrow$ spheres moves towards each other. If we do not take care of that spheres move toward each other, then we must take $u_1 = -\frac{1+\sqrt{2}}{\sqrt{2}-1} u_2$ which automatically adjusts the direction and if we already use formulas by considering that spheres move towards each other (i.e we consider $-ve$ sign in ratio 1st), then we drop the $-ve$ sign with ratio of u_1 & u_2.

Problem # 40 (m) # Two smooth equal spheres moving with velocites u_1 & u_2 in the direction of the line Joining their centres, impinge directly. If the co-efficient of restitution is $\frac{2}{3}$, then show that exactly $\frac{2}{3}\left(\frac{3-1}{3}\right)$ of the K.E is lost in collision if u_1 & u_2 are in ratio $1+\sqrt{3} : 1-\sqrt{3}$

Sol # It is given that

$$u_1 : u_2 = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$\Rightarrow \frac{u_1}{u_2} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$u_1 = \frac{1+\sqrt{3}}{1-\sqrt{3}} u_2$$

$$\text{K.E before collision} = \frac{1}{2}m(u_1^2 + u_2^2)$$

$$= \frac{1}{2}m\left[\left(\frac{1+\sqrt{3}}{1-\sqrt{3}} u_2\right)^2 + u_2^2\right]$$

$$= \frac{1}{2}m\left[\frac{(1+\sqrt{3})^2 + (1-\sqrt{3})^2}{(1-\sqrt{3})^2}\right] u_2^2 = \frac{1}{2}m \frac{2\cdot 4}{(1-\sqrt{3})^2} u_2^2$$

$$= \frac{4m}{(1-\sqrt{3})^2} u_2^2 \longrightarrow ①$$

$$K.E \text{ lost} = \frac{1}{2}\cdot\frac{m^2}{2m}\left(1-\frac{1}{9}\right)(u_1-u_2)^2$$

$$= \frac{1}{4} m \left(\frac{8}{9}\right)\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}u_2 - u_2\right)^2$$

$$= \frac{2}{9} m \cdot \left(\frac{1+\sqrt{3} - (1-\sqrt{3})}{(1-\sqrt{3})^2}\right)^2 u_2^2$$

$$= \frac{2}{9} m \cdot \frac{(2\sqrt{3})^2}{(1-\sqrt{3})^2} u_2^2$$

$$= \frac{2}{9} m \cdot \frac{4\cdot 3}{(1-\sqrt{3})^2} u_2^2$$

$$= \frac{2}{3}\left(\frac{4m}{(1-\sqrt{3})^2} u_2^2\right)$$

$$= \frac{2}{3}(\text{original } K.E) \quad (\text{Proved})$$

Problem # 41 # (m) Two smooth equal spheres moving with velocities u_1 & u_2 in the direction of line joining their centres, impinge directly. If the Co-efficient of restitution is $\frac{1}{5}$ and u_1, u_2 are in ratio $u_1 : u_2 = 1+\sqrt{5} : 1-\sqrt{5}$. Prove that exactly $4/5$ of the K.E is lost.

Sol # $u_1 : u_2 = 1+\sqrt{5} : 1-\sqrt{5}$

$$\Rightarrow \frac{u_1}{u_2} = \frac{1+\sqrt{5}}{1-\sqrt{5}}$$

$$\Rightarrow u_1 = \frac{1+\sqrt{5}}{1-\sqrt{5}} u_2$$

K.E before collision $= \frac{1}{2}m[u_1^2+u_2^2]$

$$= \frac{1}{2}m\left[\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}u_2\right)^2 + u_2^2\right]$$

$$= \frac{1}{2}m\left[\frac{(1+\sqrt{5})^2+(1-\sqrt{5})^2}{(1+\sqrt{5})^2}\right]u_2^2$$

$$= \frac{1}{\cancel{2}}m\left[\frac{\cancel{2}\cdot 6}{(1-\sqrt{5})^2}\right]u_2^2$$

$$= \frac{6m}{(1-\sqrt{5})^2}u_2^2 \longrightarrow (1)$$

$$K.E \text{ lost} = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$$

$$= \frac{1}{2}\frac{m^2}{2m}\left[1-\frac{1}{25}\right]\left[\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}u_2\right)-u_2\right]^2$$

$$= \frac{1}{4}\left(\frac{24}{25}\right)m\left[\frac{1+\sqrt{5}-(1-\sqrt{5})}{1-\sqrt{5}}\right]^2u_2^2$$

$$= \frac{6}{25}m\left[\frac{2\sqrt{5}}{1-\sqrt{5}}\right]^2u_2^2$$

$$= \frac{6}{25}\cdot\frac{4\cdot 5m}{(1-\sqrt{5})^2}u_2^2$$

$$= \frac{4}{5}\left[\frac{6m}{(1-\sqrt{5})^2}u_2^2\right]$$

$$= \frac{4}{5}(K.E \text{ before collision})$$

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Problem#42# Two smooth equal spheres moving with velocities u_1 & u_2 in the direction of the line joining their centres, impinge directly. If the co-efficient of restitution is $\frac{1}{n}$ (where n is an odd +ve integer greater than 1), then show that exactly $\frac{n-1}{n}$ of the K.E is lost in collision if $u_1 : u_2 = 1+\sqrt{n} : 1-\sqrt{n}$

Sol# It is given that

$$\frac{u_1}{u_2} = \frac{1+\sqrt{n}}{1-\sqrt{n}}$$

$$\Rightarrow u_1 = \frac{1+\sqrt{n}}{1-\sqrt{n}} u_2$$

K.E before collision $= \frac{1}{2} m (u_1^2 + u_2^2)$

$$= \frac{1}{2} m \left[\left(\frac{1+\sqrt{n}}{1-\sqrt{n}} u_2\right)^2 + u_2^2 \right]$$

$$= \frac{1}{2} m \left[\frac{(1+\sqrt{n})^2 + (1-\sqrt{n})^2}{(1-\sqrt{n})^2} \right] u_2^2$$

$$= \frac{1}{2} m \cdot \frac{2(n+1)}{(1-\sqrt{n})^2} u_2^2$$

$$= \frac{m(n+1)}{(1-\sqrt{n})^2} u_2^2 .$$

K.E lost $= \frac{1}{2} \frac{m_1 m_1}{m_1+m_2} (1-e^2)(u_1-u_2)^2$

$$= \frac{1}{2} \cdot \frac{m^2}{2m} \left[1 - \frac{1}{n^2}\right]\left[\left(\frac{1+\sqrt{n}}{1-\sqrt{n}} u_2\right)^2 + u_2\right]$$

$$= \frac{1}{4} m \left[\frac{n^2-1}{n^2}\right]\left[\frac{(1+\sqrt{n}) - (1-\sqrt{n})}{1-\sqrt{n}}\right]^2 u_2^2$$

$$= \frac{1}{4} m \frac{(2\sqrt{n})^2}{(1-\sqrt{n})^2} \cdot \left(\frac{n^2-1}{n^2}\right) u_2^2$$

$$= \frac{1}{\cancel{4}} m \cdot \frac{\cancel{4} \cdot n}{(1-\sqrt{n})^2} \cdot \frac{(n-1)(n+1)}{n^2} u_2^2$$

$$= \left(\frac{n-1}{n}\right) \left[\frac{m(n+1)}{(1-\sqrt{n})^2} \right] u_2^2$$

$$= \left(\frac{n-1}{n}\right) [\text{K.E before collision.}]$$